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Stability and Bifurcation of Equilibrium Solutions of Reaction–Diffusion Equations on an Infinite Space Interval

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Under the condition that $f(x, y, z, \alpha)$ and its partial derivatives decay sufficiently fast as $|x| \to \infty$ we will study the (linear) stability and bifurcation of equilibrium solutions of the scalar problem

$$u_t = u_{xx} + f(x, u, u_x, \alpha), \; u_x(-\infty, t) = u_x(\infty, t) = 0, \tag{$*$}$$

where α is a real bifurcation parameter. After introducing appropriate function spaces X and Y the problem $(*)$ can be rewritten

$$\frac{d}{dt} u = G(u, \alpha), \tag{$**$}$$

where $G: X \times \mathbb{R} \to Y$ is given by $G(u, \alpha)(x) = u''(x) + f(x, u(x), u'(x), \alpha)$. It will be shown, for each $(u, \alpha) \in X \times \mathbb{R}$, that the Fréchet derivative $G_u(u, \alpha): X \to Y$ is not a Fredholm operator. This difficulty is due to the fact that the domain of the space variable x is infinite and cannot be eliminated by making another choice of X and Y. Since $G_u(u, \alpha)$ is not Fredholm, the hypotheses of most of the general stability and bifurcation results are not satisfied. If $(u_0, \alpha_0) \in S = \{(u, \alpha): G(u, \alpha) = 0\}$, (i.e., (u_0, α_0) is an equilibrium solution of $(**)$), a necessary condition on the spectrum of $G_u(u_0, \alpha_0)$ for a change in the stability of points in S to occur at (u_0, α_0) will be given. When this condition is met, the principle of exchange of stability which means, in a neighborhood of (u_0, α_0), that adjacent equilibrium solutions for the same α have opposite stability properties in a weakened sense will be established. Also, when G_u or its first order partial derivatives, evaluated at (u_0, α_0), are not too degenerate, the shape of S in a neighborhood of (u_0, α_0) will be described and a strenghtened form of the principle of exchange of stability will be obtained.

1. Statement and Discussion of Results

In this paper we will be concerned with how the (linear) stability of equilibrium solutions of the scalar problem

$$u_t = u_{xx} + f(x, u, u_x, \alpha), \tag{1.1a}$$

$$u_x(-\infty, t) = u_x(\infty, t) = 0 \tag{1.1b}$$

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changes with the real parameter α under the condition that $f(x, y, z, \alpha)$ and its partial derivatives with respect to y, z, and α decay sufficiently fast as $|x|$ tends to ∞. More precisely, *we will assume throughout this paper that the following hypothesis* (H_1) *is satisfied by* f:

(a) $f: D \times \bar{J} \to \mathbb{R}$ *is continuous, where* J *is an open bounded interval of real numbers and* D *is an open subset of* $\mathbb{R}^3$ *which contains* $\{(x, y, z): |y| \leqslant m|x|$ *and* $|z| \leqslant m\}$ *for some positive number* m;

(b) *the first order partial derivatives* f_y, f_z, *and* f_α *exist and are continuous on* $D \times \bar{J}$;

(c) $\int_{-\infty}^{\infty} |x| F(x)\, dx < \infty$, *where*

$$F(x) = \max\{(|f| + |f_y| + |f_z| + |f_\alpha|)(x, y, z, \alpha): |y| \leqslant m|x|,\quad |z| \leqslant m, \text{ and } \alpha \in \bar{J}\};$$

and

(d) *for each* $\alpha_0 \in J$,

$$\lim_{|x| \to \infty} \max\{(|f| + |f_y|)(x, y, z, \alpha_0): |y| \leqslant m|x| \text{ and } |z| \leqslant m\} = 0$$

and

$$\max\{|f_z(x, y, z, \alpha_0)|: |y| \leqslant m|x| \text{ and } |z| \leqslant m\},$$

as a function of x, *is bounded.*

The more general problem

$$v_t = c^2 v_{\xi\xi} + g(\xi, v, v_\xi, \alpha),\ v_\xi(-\infty, t) = a, \qquad v_\xi(\infty, t) = b,$$

where a, b, and c are given real numbers with $c > 0$, which arises in combustion theory (see the example preceding Theorem 3), can be transformed into (1.1a), (1.1b) under the change of variables $u(x, t) = v(\xi, t) - \nu(\xi)$, $\xi = cx$, where $\nu: \mathbb{R} \to \mathbb{R}$ is any twice continuously differentiable function with $\nu(\xi) = a\xi$ for $\xi \leqslant -1$ and $\nu(\xi) = b\xi$ for $\xi \geqslant 1$. Under this change of variables the relationship between f and g is given by

$$f(x, y, z, \alpha) = c^2 \nu''(cx) + g\left(cx, y + \nu(cx), \frac{1}{c} z + \nu'(cx), \alpha\right).$$

Other examples of reaction–diffusion equations on an infinite space interval, which arise in combustion theory, can be found in [1, 9]. Since, in all these examples, the space domain is infinite, the linearized equation is not a Fredholm operator and its spectrum may have a continuous as well as a discrete part. In the past most investigators have ignored the continuous spectrum, assuming it to be in the left half plane, and then resorted to numerical and/or ad hoc analytic methods for finding the discrete spectrum. Here we give a systematic and rigorous treatment of a fairly general class of equations, which takes into account both the continuous and discrete spectrum.

Let (H_2) *be the hypothesis that* (H_1) *holds, all the second order partial derivatives of f with respect to y, z, and α exist and are continuous on* $D \times \bar{J}$, *and* $\int_{-\infty}^{\infty} |x|\, G(x)\, dx < \infty$, *where*

$$G(x) = \max\{(x^2 |f_{yy}| + |x|\,|f_{yz}| + |x|\,|f_{y\alpha}| + |f_{zz}| + |f_{\alpha\alpha}| + |f_{z\alpha}|) \\ (x, y, z, \alpha): |y| \leqslant m\,|x|, |z| \leqslant m, \text{ and } \alpha \in \bar{J}\}.$$

To state what kind of perturbations will be used for determining the stability of equilibrium solutions of (1.1a) (1.1b) we introduce the following spaces. Let X be the complex vector space of twice continuously differentiable functions $u: \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\infty} u(x)$ and $\lim_{x\to-\infty} u(x)$ both exist and are finite, and

$$0 = \lim_{|x|\to\infty} u'(x) = \lim_{|x|\to\infty} u''(x).$$

The norm

$$\|u\|_X = \sup_{x\in\mathbb{R}} |u(x)| + \max_{x\in\mathbb{R}} |u'(x)| + \max_{x\in\mathbb{R}} |u''(x)|$$

makes X a complex Banach space. Let Y be the complex vector space of continuous functions $u: \mathbb{R} \to \mathbb{C}$ such that $\lim_{x\to\infty} u(x)$ and $\lim_{x\to-\infty} u(x)$ both exist and are finite. The norm $\|u\|_Y = \sup_{x\in\mathbb{R}} |u(x)|$ makes Y a complex Banach space. Let $\hat{X}$ and $\hat{Y}$ be the subspaces of X and Y, respectively, consisting of real valued functions. Let $\mathscr{O}$ be the open subset of $\hat{X} \times \mathbb{R}$ given by

$$\mathscr{O} = \{(u, \alpha): u \in \hat{X} \text{ and } (x, u(x), u'(x), \alpha) \in D \times J \text{ for all } x \in \mathbb{R}\}.$$

We can now restate (1.1a), (1.1b) as

$$\frac{du}{dt} = G(u, \alpha), \tag{1.2}$$

where $G: \mathscr{O} \to \hat{Y}$ is given by

$$G(u, \alpha)(x) = u''(x) + f(x, u(x), u'(x), \alpha).$$

The equilibrium solutions of (1.2) are given by solutions, in $\mathscr{O}$, of

$$G(u, \alpha) = 0. \tag{1.3}$$

Because of hypothesis (H_1), if $\alpha \in J$ and $y(x)$ is a solution of $y'' + f(x, y, y', \alpha) = 0$ with $\lim_{|x| \to \infty} y'(x) = 0$ then $y \in X$, and thus the restriction that equilibrium solutions be in $\mathcal{O}$ is really no restriction at all.

To describe our results and define what it means for an equilibrium solution of (1.2) to be stable, we state three propositions, which will be proved in Section 3. The notation introduced in Propositions 1 and 3 will be used throughout this paper.

PROPOSITION 1. *Suppose the hypothesis* (H_1) *holds. For each* $(\alpha, \beta) \in J \times \mathbb{R}$ *the initial value problem*

$$y'' + f(x, y, y', \alpha) = 0, \tag{1.4a}$$

$$\lim_{x \to -\infty} y'(x) = 0, \qquad \lim_{x \to -\infty} y(x) = \beta \tag{1.4b}$$

has a unique solution, $y(x, \alpha, \beta)$, *defined in a neighborhood of* $-\infty$. *The set* Ω *of all* $(\alpha, \beta) \in J \times \mathbb{R}$ *such that* $y(x, \alpha, \beta)$ *exists on* $-\infty < x < \infty$ *and* $y'(\infty, \alpha, \beta) = \lim_{x \to \infty} y'(x, \alpha, \beta)$ *exists and is in absolute value less than m, is open. The function* $y'(\infty, \alpha, \beta)$ *is continuously differentiable on* Ω, $y(x, \alpha, \beta)$ *and* $y'(x, \alpha, \beta)$ *are continuously differentiable on* $\mathbb{R} \times \Omega$, *and, for each* $(\alpha, \beta) \in \Omega$, $y_\alpha(x, \alpha, \beta)$ *and* $y_\beta(x, \alpha, \beta)$, *as functions of x, satisfy*

$$w'' + f_y w + f_z w' + f_\alpha = 0, \tag{1.5a}$$

$$\lim_{x \to -\infty} w(x) = \lim_{x \to -\infty} w'(x) = 0, \tag{1.5b}$$

and

$$w'' + f_y w + f_z w' = 0, \tag{1.6a}$$

$$\lim_{x \to -\infty} w'(x) = 0, \qquad \lim_{x \to -\infty} w(x) = 1, \tag{1.6b}$$

respectively, where the partial derivatives of f are evaluated at $(x, y(x, \alpha, \beta), y'(x, \alpha, \beta), \alpha)$. *For* $(\alpha, \beta) \in \Omega$,

$$\lim_{x \to \infty} \frac{\partial}{\partial x} y_\alpha(x, \alpha, \beta) = \frac{\partial}{\partial \alpha} y'(\infty, \alpha, \beta)$$

and

$$\lim_{x \to \infty} \frac{\partial}{\partial x} y_\beta(x, \alpha, \beta) = \frac{\partial}{\partial \beta} y'(\infty, \alpha, \beta). \tag{1.7}$$

If $(\alpha_0, \beta_0) \in \Omega$ *then there is a continuous function* $F_0(x)$ *and* $\delta > 0$ *such that* $\int_{-\infty}^{\infty} |x| F_0(x)\, dx < \infty$ *and* $|f| + |f_y| + |f_z| + |f_\alpha|$ *evaluated at* $(x, y(x, \alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta), y'(x, \alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta), \alpha_0 + \Delta\alpha)$ *is less than* $F_0(x)$ *for all* x *and* $|\Delta\alpha| + |\Delta\beta| < \delta$.

Furthermore, if hypothesis (H_2) *holds then* $y'(\infty, \alpha, \beta)$ *is twice continuously differentiable on* Ω, *the second order partial derivatives of* $y(x, \alpha, \beta)$ *and* $y'(x, \alpha, \beta)$ *with respect to* α *and* β *exist and are continuous on* $\mathbb{R} \times \Omega$, *and, for each* $(\alpha, \beta) \in \Omega$, $y_{\alpha\alpha}(x, \alpha, \beta)$, $y_{\alpha\beta}(x, \alpha, \beta)$, *and* $y_{\beta\beta}(x, \alpha, \beta)$, *as functions of* x, *satisfy*

$$\begin{aligned} w'' + f_y w + f_z w' + f_{yy} y_\alpha^2 + f_{zz} y_\alpha'^2 + f_{\alpha\alpha} \\ + 2f_{yz} y_\alpha y_\alpha' + 2f_{y\alpha} y_\alpha + 2f_{z\alpha} y_\alpha' = 0, \end{aligned} \tag{1.8a}$$

$$\lim_{x \to -\infty} w(x) = \lim_{x \to -\infty} w'(x) = 0, \tag{1.8b}$$

$$\begin{aligned} w'' + f_y w + f_z w' + f_{yy} y_\alpha y_\beta + f_{zz} y_\alpha' y_\beta' \\ + f_{yz}(y_\alpha y_\beta' + y_\alpha' y_\beta) + f_{y\alpha} y_\beta + f_{z\alpha} y_\beta' = 0, \end{aligned} \tag{1.9a}$$

$$\lim_{x \to -\infty} w(x) = \lim_{x \to -\infty} w'(x) = 0, \tag{1.9b}$$

and

$$w'' + f_y w + f_z w' + f_{yy} y_\beta^2 + f_{zz} y_\beta'^2 + 2f_{yz} y_\beta y_\beta' = 0, \tag{1.10a}$$

$$\lim_{x \to -\infty} w(x) = \lim_{x \to -\infty} w'(x) = 0, \tag{1.10b}$$

respectively, where the partial derivatives of f *are evaluated at* $(x, y(x, \alpha, \beta), y'(x, \alpha, \beta), \alpha)$ *and the partial derivatives of* y *are evaluated at* (x, α, β). *Also*

$$\lim_{x \to \infty} \frac{\partial}{\partial x} [y_{\alpha\alpha}(x, \alpha, \beta)] = \frac{\partial^2}{\partial \alpha^2} y'(\infty, \alpha, \beta) \tag{1.11}$$

and similarly for $y_{\alpha\beta}$ *and* $y_{\beta\beta}$.

Let S be the set of all $(\alpha, \beta) \in \Omega$ such that $y'(\infty, \alpha, \beta) = 0$. Since $y'(\infty, \alpha, \beta)$ is continuous on Ω, S is a closed subset of Ω. If $\alpha \in J$, then (u, α) is a solution in $\mathscr{O}$ of (1.3) if and only if, for some $\beta \in$?, we have $(\alpha, \beta) \in S$ and $u(x) = y(x, \alpha, \beta)$. Thus we have a one to one correspondence between equilibrium solutions (u, α) in $\mathscr{O}$ and points (α, β) in S.

DEFINITION 1. If (H_1) holds and $(\alpha, \beta) \in \Omega$ then $L(\alpha, \beta)$ is the linear operator from X into Y given by

$$L(\alpha, \beta)\, \varphi = \varphi'' + f_y \varphi + f_z \varphi',$$

where the partial derivatives of f are evaluated at $(x, y(x, \alpha, \beta), y'(x, \alpha, \beta), \alpha)$. The spectrum of $L(\alpha, \beta)$ is $\{\zeta \in \mathbb{C}: L(\alpha, \beta) - \zeta I$ is not an isomorphism of X onto $Y\}$, where I is the inclusion of X into Y.

Note that if $(\alpha, \beta) \in S$ and (u, α) is its corresponding point in $\mathscr{C}$ then $L(\alpha, \beta) = G_u(u, \alpha)$.

For $(\alpha, \beta) \in \Omega$, a useful description of the spectrum of $L(\alpha, \beta)$ is provided by the following two propositions.

PROPOSITION 2. *Suppose hypothesis* (H_1) *holds and* $(\alpha, \beta) \in \Omega$. *Let* $L = L(\alpha, \beta)$. *Then the spectrum of* L, *denoted by* Σ, *is a subset of the real axis*, $\Sigma \cap (0, \infty)$ *is a set bounded above by* $\|f_y(\cdot, y(\cdot, \alpha, \beta), y'(\cdot, \alpha, \beta), \alpha)\|_Y$ *with zero as its only possible cluster point, and the codimension of the range of* L *in* Y *is infinite, (and thus the supremum of* Σ *belongs to* Σ *and is nonnegative). If* $\zeta \in \Sigma \cap (0, \infty)$ *then* $L - \zeta I$ *has a one-dimensional null space. If* λ *is a nonnegative eigenvalue of* L *then* $\lambda = \max \Sigma$ *if and only if* $L - \lambda I$ *has a one-dimensional null space spanned by a positive function* $\varphi \in \hat{X}$.

By Proposition 2, zero is in the spectrum of $L(\alpha, \beta)$ for all $(\alpha, \beta) \in \Omega$. In the case the maximum of the spectrum of $L(\alpha, \beta)$ is zero the following proposition gives a necessary and sufficient condition for zero to be an eigenvalue of $L(\alpha, \beta)$.

PROPOSITION 3. *Suppose the hypothesis* (H_1) *holds. For* $(\alpha, \beta) \in \Omega$, *let* $\lambda(\alpha, \beta)$ *be the maximum of the spectrum of* $L(\alpha, \beta)$. *Then* $\lambda(\alpha, \beta)$ *is nonnegative and continuous on* Ω. *Suppose* $\lambda(\alpha_0, \beta_0) = 0$. *If* $y'_\beta(\infty, \alpha_0, \beta_0) \neq 0$ *then for all* (α, β) *in some neighborhood of* (α_0, β_0) *we have* $\lambda(\alpha, \beta) = 0$ *and zero is not an eigenvalue of* $L(\alpha, \beta)$. *If* $y'_\beta(\infty, \alpha_0, \beta_0) = 0$ *then* $L(\alpha_0, \beta_0)$ *has a one-dimensional null space spanned by a positive function* $\varphi_0 \in \hat{X}$ *such that* $\varphi_0(-\infty) = 1$ *and* $\varphi'_0(-\infty) = 0$.

By Proposition 3 and the implicit function theorem, if $(\alpha_1, \beta_1) \in S$ and $y'_\beta(\infty, \alpha_1, \beta_1) \neq 0$ then, in some neighborhood V of (α_1, β_1), S is given by the graph of a continuously differentiable function $\beta = \beta(\alpha)$, and $\lambda(\alpha, \beta)$ is either identically zero in V or never zero in V. Thus studying S in a neighborhood of a point (α_1, β_1) with $y'_\beta(\infty, \alpha_1, \beta_1) \neq 0$ provides very little information about the topological structure of S; and if (α_0, β_0) is a point of S at which $\operatorname{sgn} \lambda(\alpha, \beta)$ changes (i.e., every neighborhood of (α_0, β_0) contains points where $\lambda > 0$ and points where $\lambda = 0$) then $\lambda(\alpha_0, \beta_0) = 0$ and $y'_\beta(\infty, \alpha_0, \beta_0) = 0$, (or equivalently, by Proposition 3, $\lambda(\alpha_0, \beta_0) = 0$ and zero is an eigenvalue of $L(\alpha_0, \beta_0)$). Accordingly, we will only study S and λ in a neighborhood of those points $(\alpha_0, \beta_0) \in S$ for which $\lambda(\alpha_0, \beta_0) = 0$ and zero is an eigenvalue of $L(\alpha_0, \beta_0)$.

DEFINITION 2. A point $(\alpha,\beta)\in S$ is

(i) a critical point if $\lambda(\alpha,\beta)=0$ and zero is an eigenvalue of $L(\alpha,\beta)$:

(ii) a stable point if $\lambda(\alpha,\beta)=0$ and zero is not an eigenvalue of $L(\alpha,\beta)$; and

(iii) an unstable point if $\lambda(\alpha,\beta)>0$.

Thus each point of S, (and hence each equilibrium solution of (1.2)), has exactly one of three stability properties of Definition 2.

By Proposition 2, if $(u,\alpha)\in\mathscr{E}$ is an equilibrium solution of (1.2) corresponding to the point $(\alpha,\beta)\in S$ then $L(\alpha,\beta): X\to Y$ is not a Fredholm operator. This difficulty is due to the fact that the domain of the space variable x is infinite and cannot be eliminated by making another choice for X and Y. Since $G_u(u,\alpha)=L(\alpha,\beta)$ is not Fredholm, G does not satisfy the hypotheses of the general stability and bifurcation theorems which appear in Crandall and Rabinowitz [5] and Weinberger [12]. However, we will show that the conclusions obtained in [5] and [12] are valid for our functional G. In fact, taken together, the following proposition and theorem form, for our functional G, the analog of the results in [12, Sect. 2]. Theorem 1 is sometimes called the principle of exchange of stability.

PROPOSITION 4. *Suppose hypothesis* (H_1) *holds and* (α_0,β_0) *is a critical point of* S. *Then there is a neighborhood* V *of* (α_0,β_0) *such that for* $(\alpha,\beta)\in V\cap S$ *we have* $y'_\beta(\infty,\alpha,\beta)>(=,<)\,0$ *if and only if* (α,β) *is a stable (critical, unstable) point of* S.

THEOREM 1. *Suppose hypothesis* (H_1) *holds and* (α_0,β_0) *is a critical point of* S. *Then there is neighborhood* V *of* (α_0,β_0) *such that if* $\gamma(s)=(\alpha(s),\beta(s))$, $0\leqslant s\leqslant 1$, *is a differentiable curve in* V *with* $\alpha'(0)=\alpha'(1)=0$, $\gamma(0)$ *and* $\gamma(1)$ *in* S, *and* $\gamma(s)$ *not in* S *for* $0<s<1$ *then*

(i) $\beta'(0)\beta'(1)>0$ *implies* $\gamma(0)$ *and* $\gamma(1)$ *are not both stable (unstable)*;

(ii) $\beta'(0)\beta'(1)<0$ *implies it is not the case that* $\gamma(0)$ *is stable (unstable) and* $\gamma(1)$ *is unstable (stable)*.

If (α_0,β_0) is a critical point of S, in Theorems 2 and 3 we will describe the possible shapes of S in a neighborhood of (α_0,β_0) and obtain a strengthened form of the principle of exchange of stability provided either $\nabla_{(\alpha,\beta)}y'(\infty,\alpha_0,\beta_0)\neq(0,0)$ or, in the case hypothesis (H_2) holds, the determinant of the 2×2 Hessian $D^2y'(\infty,\alpha_0,\beta_0)$ is not zero. These two theorems form the analog of the results in [5] for our functional G.

Throughout this paper we will implicitly and repeatedly use the fact (see

[4 Chap. 3, Problems 29 and 35]) that if $\int_{-\infty}^{\infty} |xg(x)|\,dx < \infty$, $\int_{-\infty}^{\infty} |xh(x)|\,dx < \infty$, and $\lambda \geqslant 0$ then the general solution of

$$w'' + g(x)\,w + h(x)\,w' = \lambda^2 w \tag{1.12}$$

is given by $w = c_1 w_1 + c_2 w_2$, where c_1 and c_2 are constants and w_1 and w_2 are solutions of (1.12) satisfying as $x \to -\infty$, $w_1(x) = e^{\lambda x}(1 + o(1))$,

$$\begin{aligned}
w_1'(x) &= \lambda e^{\lambda x}(1 + o(1)), && \text{if} \quad \lambda > 0,\\
&= o\left(\frac{1}{x}\right), && \text{if} \quad \lambda = 0;\\
w_2(x) &= e^{-\lambda x}(1 + o(1)), && \text{if} \quad \lambda > 0,\\
&= x(1 + o(1)), && \text{if} \quad \lambda = 0;\\
w_2'(x) &= -\lambda e^{-\lambda x}(1 + o(1)), && \text{if} \quad \lambda > 0,\\
&= 1 + o(1), && \text{if} \quad \lambda = 0.
\end{aligned}$$

Of course we have a similar statement concerning the behavior of solutions of (1.12) in a neighborhood of $+\infty$.

Also if $y(x)$ and $w(x)$ satisfy

$$y'' + g_1(x)\,y + h_1(x)\,y' = k(x)$$

and

$$w'' + g_2(x)\,w + h_2(x)\,w' = l(x),$$

respectively, and $H_i(x) = \exp(\int^x h_i)$, then Picone's formula (see [6]) for $y(x)$ and $w(x)$ is

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{y}{w}\,(H_1 y'w - H_2 yw')\right] + (H_1 g_1 - H_2 g_2)\,y^2\\
&\qquad = (H_1 - H_2)(y')^2 + H_2\left(\frac{y'w - yw'}{w}\right)^2 + \frac{y}{w}\,(kH_1 w - lH_2 y)
\end{aligned}$$

and Green's formula for $y(x)$ and $w(x)$ is

$$\begin{aligned}
&\frac{d}{dx}\,[y'wH_1 - yw'H_2] + (g_1 H_1 - g_2 H_2)\,yw\\
&\qquad = (H_1 - H_2)\,y'w' + (kwH_1 - lyH_2).
\end{aligned}$$

Before stating Theorems 2 and 3 we make one further observation. Let (α_0, β_0) be a critical point of S. Then, by Proposition 3, $y'_\beta(\infty, \alpha_0, \beta_0) = 0$ and the solution, $\varphi_0(x)$, of the problem

$$\varphi'' + f_y^0 \varphi + f_z^0 \varphi' = 0, \tag{1.13a}$$

$$\varphi(-\infty) = 1, \qquad \varphi'(-\infty) = 0 \tag{1.13b}$$

(where here and throughout this paper a superscript 0 on a partial derivative of f will stand for the function of the single variable x obtained by evaluating that partial derivative at $(x, y(x, \alpha_0, \beta_0), y'(x, \alpha_0, \beta_0), \alpha_0))$ is positive and in X, and hence $\lim_{x \to \infty} \varphi_0(x) = \varphi_0(\infty)$ exists and is finite and positive. (If $\varphi_0(\infty) = 0$ then $\varphi_0 \equiv 0$.) Green's formula for $y_\alpha(x, \alpha_0, \beta_0)$ and $\varphi_0(x)$, (remember $y_\alpha(x, \alpha_0, \beta_0)$ satisfies (1.5a, b) with $(\alpha, \beta) = (\alpha_0, \beta_0)$), is

$$y'_\alpha(\infty, \alpha_0, \beta_0)\, \varphi_0(\infty) \exp\left(\int_{-\infty}^{\infty} f_z^0\right) = -\int_{-\infty}^{\infty} f_\alpha^0\, \varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0\right) dx.$$

(Here we have used the fact, which follows from (1.7), that $y_\alpha(x, \alpha_0, \beta_0) = \mathcal{O}(x)$ as $x \to \infty$.) Thus $\nabla_{(\alpha,\beta)} y'(\infty, \alpha_0, \beta_0) \neq (0, 0)$ if and only if $y'_\alpha(\infty, \alpha_0, \beta_0) \neq 0$ if and only if

$$\int_{-\infty}^{\infty} f_\alpha^0\, \varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0\right) dx \tag{1.14}$$

is not zero. Theorem 2 will deal with the case that (1.14) is not zero, and Theorem 3 (in the case hypothesis (H_2) holds), with the case that (1.14) is zero and the determinant of $D^2 y'(\infty, \alpha_0, \beta_0)$ is nonzero.

THEOREM 2. *Suppose hypothesis* (H_1) *holds and* (α_0, β_0) *is a critical point of S. Let* $\varphi_0(x)$ *be the solution of* (1.13a) (1.13b) *and suppose* (1.14) *is not zero. Then, in a neighborhood of* (α_0, β_0), *S is given by the graph of a continuously differentiable function* $\alpha = \alpha(\beta)$ *defined in a neighborhood of* $\beta = \beta_0$ *with* $\alpha(\beta_0) = \alpha_0$ *and* $\alpha'(\beta_0) = 0$. *There is an open interval K containing* β_0, *such that for* $\beta \in K$ *we have* $(\alpha(\beta), \beta)$ *is a stable (critical, unstable) point if and only if*

$$\alpha'(\beta) \int_{-\infty}^{\infty} f_\alpha^0\, \varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0\right) dx > (=, <)\ 0. \tag{1.15}$$

Furthermore, if the second order partial derivatives of f with respect to y, z, and α *exist and are continuous on* $D \times \bar{J}$, $\int_{-\infty}^{\infty} |x|\, H(x)\, dx < \infty$, *where*

$$H(x) = \max\{(|f_{yy}| + |f_{zz}| + |f_{yz}| + |f_{y\alpha}| + |f_{z\alpha}|)(x, y, z, \alpha): \\ |y| \leqslant m\,|x|, |z| \leqslant m, \alpha \in \bar{J}\}$$

and $E \neq 0$, *where*

$$E = \int_{-\infty}^{\infty} \left[f_{yy}^{0} \varphi_0 + f_{yz}^{0} \varphi_0' + f_y^0 \int_{-\infty}^{x} (f_{yz}^{0} \varphi_0 + f_{zz}^{0} \varphi_0')\, dt \right] \varphi_0^2 \exp\left(\int_{-\infty}^{x} f_z^0\, dt \right) dx$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{x} (f_{yz}^{0} \varphi_0 + f_{zz}^{0} \varphi_0')\, dt \varphi_0'(x)^2 \exp\left(\int_{-\infty}^{x} f_z^0\, dt \right) dx, \tag{1.16}$$

then, in some neighborhood of $\beta = \beta_0$, *we have*

$$\operatorname{sgn} \alpha'(\beta) = -\operatorname{sgn} \left[(\beta - \beta_0)\, E \Big/ \int_{\infty}^{\infty} f_\alpha^0 \varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0 \right) dx \right] \tag{1.17}$$

and hence the inequality (1.15) *is equivalent to*

$$(\beta - \beta_0)\, E < (=, >)\, 0.$$

Remark 1. Note that if the hypotheses of the first paragraph of Theorem 2 are satisfied (these hypotheses are just the hypotheses of Theorem 1 together with the condition that (1.14) is not zero) then the last sentence of the first paragraph of Theorem 2 contains as a special case the conclusion of Theorem 1 and in addition says, for points (α, β) in S which are sufficiently close to (α_0, β_0), that (α, β) is a critical point of S if and only if the tangent line to S at (α, β) is vertical. This addition to the conclusion of Theorem 1 can be viewed as a strengthened form of the principle of exchange of stability.

Remark 2. Note that if u_x does not appear on the right side of (1.1a) (i.e., f is only a function of x, y, and α) then the formula for E, in Theorem 2, reduces to $\int_{-\infty}^{\infty} f_{yy}^0 \varphi_0^3\, dx$.

Remark 3. Since $\varphi_0(x)$ is positive for all x. If f_α is positive (negative) on $D \times J$, then (1.14) is positive (negative). Similarly, if u_x does not appear on the right side of (1.1a) and f_{yy} is positive (negative) on $D \times J$, then E is positive (negative).

Example. The problem of finding the steady states and exactly where the stability of these steady states changes, for the boundary value problem

$$v_t = v_{xx} + (x^2 - v^2)\, e^{-\alpha(v + \gamma x)}, \tag{1.18a}$$

$$v_x(-\infty, t) = -1, \qquad v_x(\infty, t) = 1. \tag{1.18b}$$

where γ is a fixed constant with $|\gamma| < 1$, and α is the real bifurcation parameter which ranges over $J_\gamma = \{\alpha: 0 < a_\gamma < \alpha < b_\gamma < \infty\}$ arises in combustion theory [7, Eqs. (75)–(77): 2, p. 117]. In [7, Figs. 8 and 9] several of the steady states are graphed for $\gamma = 0$ and $\gamma = \frac{1}{2}$. In [7, Fig. 11, here

$\delta = \alpha^{-3}]$ the response curve S_γ is graphed for several values of γ. By making the change of variables mentioned in the paragraph following hypothesis (H_1), (1.18a), (1.18b) is transformed into (1.1a), (1.1b) with

$$f(x, y, z, \alpha) = v''(x) + [x^2 - (y + v(x))^2] \, e^{-\alpha(y + v(x) + \gamma x)}.$$

Let $m = (1 - |\gamma|)/2$. Then $F(x)$ in part (c) of hypothesis (H_1) and $G(x)$ in hypothesis (H_2) are both $\mathscr{O}(\exp(-a_\gamma((1 - |\gamma|)/4) \, |x|))$ as $|x| \to \infty$. Thus $f(x, y, z, \alpha)$ satisfies hypothesis (H_2). Since the graphs, in the xv plane, of the steady states of (1.18a), (1.18b) lie strictly above the graph of $v = |x|$, we have the graphs, in the xu plane, of steady states of (1.1a), (1.1b) lie strictly above the graph of $u = |x| - v(x)$, and thus $f_\alpha(x, y(x), y'(x), \alpha) > 0$ for all x and for any steady state, $u = y(x)$, of (1.1a), (1.1b). Thus, as in Remark 3, (1.14) is positive at a critical point of S_γ. Thus, by Theorem 2, at all critical points of S_γ the tangent line to S_γ exists and is vertical; and, by (1.15), points on S_γ, in a neighborhood of a critical point, are stable (unstable) if $\alpha'(\beta) > (<) \, 0$. Thus, if the response curves S_γ are as shown in [7, Fig. 11], then each S_γ has exactly one critical point, the lower branch of S_γ is stable (recall $\delta = \alpha^{-3}$) and the upper branch is unstable. This example is taken almost verbatim from [3]. The author wishes to thank J. Buckmaster and A. Nachman for allowing him to include it here. Some of the qualitative features in the graphs [7, Figs. 8, 9, and 11], which were obtained in [7] numerically, and which are needed to make the above stability remarks complete, will be verified analytically in [3]. Also, in [3], the physical significance of the above stability remarks will be discussed and compared with the numerically obtained stability results of Peters [10].

THEOREM 3. *Suppose hypothesis* (H_2) *holds and* (α_0, β_0) *is a critical point of* S. *Let* $\varphi_0(x)$ *be the solution of* (1.13a), (1.13b), *let* $v(x)$ *be the solution of* (1.5a), (1.5b) *with* $(\alpha, \beta) = (\alpha_0, \beta_0)$, *and suppose* (1.14) *is zero. Then the three integrals*

$$A = \int_{-\infty}^{\infty} [f^0_{yy} v^2 + f^0_{zz} v'^2 + f^0_{\alpha\alpha} + 2f^0_{yz} v v' + 2f^0_{y\alpha} v + 2f^0_{z\alpha} v'] \, \varphi_0 \times \exp\left(\int_{-\infty}^{x} f^0_z\right) dx,$$

$$B = \int_{-\infty}^{\infty} [f^0_{yy} v \varphi_0^2 + f^0_{zz} v' \varphi_0' \varphi_0 + f^0_{yz}(v\varphi_0' + v'\varphi_0) \, \varphi_0 + f^0_{y\alpha} \varphi_0^2 - f^0_{z\alpha} \varphi_0' \varphi_0] \times \exp\left(\int_{-\infty}^{x} f^0_z\right) dx,$$

$$C = \int_{-\infty}^{\infty} |f_{yy}^0 \varphi_0^3 + f_{zz}^0 (\varphi_0')^2 \varphi_0 + 2 f_{yz}^0 \varphi_0^2 \varphi_0'| \exp\left(\int_{-\infty}^{x} f_z^0\right) dx$$

all converge.

If $B^2 - AC < 0$ *then* (α_0, β_0) *is an isolated point of* S.

Suppose $B^2 - AC > 0$. *Let* l_1 *and* l_2 *be the two distinct lines with* l_1 *not vertical and* l_2 *not horizontal intersecting at* (α_0, β_0), *where the quadratic form*

$$A(\alpha - \alpha_0)^2 + 2B(\alpha - \alpha_0)(\beta - \beta_0) + C(\beta - \beta_0)^2$$

vanishes. Then, for some neighborhood V *of* (α_0, β_0) *we have*

(i) $S \cap V$ *is given by the graphs,* γ_1 *and* γ_2, *of two continuously differentiable functions,* $\beta = \beta_1(\alpha)$, α *in a neighborhood of* $\alpha = \alpha_0$, *and* $\alpha = \alpha_2(\beta)$, β *in a neighborhood of* $\beta = \beta_0$, *respectively, with* γ_i *tangent to* l_i *at* (α_0, β_0) *for* $i = 1, 2$;

(ii) *for* $(\alpha, \beta) \in (\gamma_1 \cup \gamma_2) - \{(\alpha_0, \beta_0)\}$, *we have, in addition to the conclusion of Theorem* 1, *that* (α, β) *is a critical point of* S *if and only if the tangent line to* S *at* (α, β) *is vertical*; *and*

(iii) *if* $f_z(x, y, z, \alpha) \equiv 0$ *in* $D \times J$ *then* $B + C\beta_1'(\alpha_0) \neq 0$ *and* (α, β) *is a stable (critical, unstable) point of* γ_1 *if and only if*

$$(\alpha - \alpha_0)(B + C\beta_1'(\alpha_0)) < (=, >)\, 0. \tag{1.19}$$

Remark 5. It will be shown in the proof of Theorem 3 that the determinant of the 2×2 Hessian, $D^2 y'(\infty, \alpha_0, \beta_0)$, is not zero if and only if $B^2 - AC \neq 0$.

Remark 6. If, as assumed in part (iii) of Theorem 3, $f_z(x, y, z, \alpha) \equiv 0$ in $D \times J$, then

$$A = \int_{-\infty}^{\infty} (f_{yy}^0 v^2 \varphi_0 + 2 f_{y\alpha}^0 v \varphi_0 + f_{\alpha\alpha}^0 \varphi_0)\, dx,$$

$$B = \int_{-\infty}^{\infty} (f_{yy}^0 v \varphi_0^2 + f_{y\alpha}^0 \varphi_0^2)\, dx,$$

$$C = \int_{-\infty}^{\infty} f_{yy}^0 \varphi_0^3\, dx.$$

Remark 7. If a and b are fixed real numbers with $a^2 + b^2 = 1$ then the results of this paper carry over to the problem

$$u_t = u_{xx} + f(x, u, u_x, \alpha), \qquad 0 \leqslant x < \infty,$$

$$a u(0, t) + b u_x(0, t) = 0, \qquad u_x(\infty, t) = 0.$$

To see this simply replace (1.4b) in Proposition 1 with

$$ay(0)+by'(0)=0, \qquad by(0)-ay'(0)=\beta \tag{1.4b$'$}$$

and then make straightforward corresponding changes throughout the paper. The second equation of (1.4b′) is the boundary condition which complements the first equation of (1.4b′) in the formulas of Green and Picone. (See [4, Chap. 11]).

2. Preliminary Lemmas

In this section we state and prove 5 lemmas which will be needed in Section 3 where we will establish the propositions and theorems of Section 1.

Lemma 1. *Suppose $g_n, h_n: \mathbb{R} \to \mathbb{R}$, $n=0, 1, 2,...,$ and $k: \mathbb{R} \to [0, \infty)$ are measurable functions, $\int_{-\infty}^{\infty} (|x|+1)\, k(x) < \infty$, g_n and h_n converge pointwise on $\mathbb{R}$ to g_0 and h_0, respectively, as $n \to \infty$, and for $x \in \mathbb{R}$ and n a nonnegative integer we have $|g_n(x)| \leqslant k(x)$ and $|h_n(x)| \leqslant k(x)$. Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers which converges to λ_0 as $n \to \infty$, and let $w_n(x)$ be the unique solution of*

$$w''+g_n(x)\, w+h_n(x)\, w' = \lambda_n^2 w, \tag{2.1a}$$

$$w(x)=e^{-\lambda_n |x|}(1+o(1)) \qquad \text{as} \quad x \to \infty \quad (x \to -\infty). \tag{2.1b}$$

Then, for each $b \in \mathbb{R}$, $e^{\lambda_n |x|}(w_n(x), w_n'(x))$ converges uniformly on $b \leqslant x < \infty$ $(-\infty < x \leqslant b)$, to $e^{\lambda_0 |x|}(w_0(x), w_0'(x))$ as $n \to \infty$.

If, in addition, for each positive integer n we have $w_n'(x)$ approaches zero as $x \to -\infty$ $(x \to \infty)$, then

(i) *$w_0'(x)$ approaches zero as $x \to -\infty$ $(x \to \infty)$;*

(ii) *$e^{\lambda_n |x|}(w_n(x), w_n'(x))$ converges uniformly on $-\infty < x < \infty$ to $e^{\lambda_0 |x|}(w_0(x), w_0'(x))$ as $n \to \infty$;*

(iii) *the functions $w_n(x)$ and $w_n'(x)$ are uniformly bounded on $-\infty < x < \infty$; and*

(iv) *if $w_n(x) > 0$ for $-\infty < x < \infty$ and n a positive integer then $w_0(x) > 0$ for $-\infty < x < \infty$.*

Proof. For $\lambda \geqslant 0$ and $x \geqslant 0$ let $\kappa(x, \lambda) = e^{-\lambda x} e^{-A(\lambda) x}$, where

$$A(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}.$$

Then

$$\kappa(x, \lambda) = \begin{pmatrix} \frac{1}{2}(1 + e^{-2\lambda x}) & -x\psi(2\lambda x) \\ -\frac{\lambda}{2}(1 - e^{-2\lambda x}) & \frac{1}{2}(1 + e^{-2\lambda x}) \end{pmatrix}, \tag{2.2}$$

where ψ: $[0, \infty) \to [0, \infty)$ is the decreasing, concave up, analytic function given by

$$\begin{aligned} \psi(x) &= \frac{1 - e^{-x}}{x}, & &\text{if} \quad x > 0, \\ &= 1, & &\text{if} \quad x = 0. \end{aligned}$$

Since ψ is concave up and decreasing we have for all $x_1, x_2 \in [0, \infty)$ that $|\psi(x_1) - \psi(x_2)| < \psi(0) - \psi(|x_1 - x_2|)$ and one easily checks, for $x \geqslant 0$, that $\psi(0) - \psi(x) \leqslant 1 - e^{-x}$. Thus, for $x_1, x_2 \in [0, \infty)$, we have

$$|\psi(x_1) - \psi(x_2)| \leqslant 1 - e^{-|x_1 - x_2|}. \tag{2.3}$$

Since, for $x \geqslant 0$, $\psi(x) \leqslant 1$, we have by (2.2), for $\lambda \geqslant 0$ and $x \geqslant 0$, that

$$\|\kappa(x, \lambda)\| \leqslant 2 + \lambda + x, \tag{2.4}$$

where we define the norm of a matrix as the sum of the absolute values of its entries. By (2.2) and (2.3) we have, for $x, \lambda_1, \lambda_2 \in [0, \infty)$, that

$$\|\kappa(x, \lambda_1) - \kappa(x, \lambda_2)\| \leqslant (1 - e^{-2|\lambda_2 - \lambda_1| x})(1 + \lambda_1 + \lambda_2 + x) + |\lambda_2 - \lambda_1|. \tag{2.5}$$

We will only prove the half of the lemma without parentheses. The proof of the other half is similar.

Rewriting (2.1a) as

$$y' = (A_n - G_n(x))\, y, \tag{2.6}$$

where $A_n = A(\lambda_n)$ and

$$G_n(x) = \begin{pmatrix} 0 & 0 \\ g_n(x) & h_n(x) \end{pmatrix}$$

and applying variation of parameters to (2.6) we get

$$y_n(x) = e^{-\lambda_n x}(1, -\lambda_n)^T + \int_x^{\infty} e^{-A_n(\xi - x)}\, G_n(\xi)\, y_n(\xi)\, d\xi, \tag{2.7}$$

where $y_n(x) = (w_n(x), w_n'(x))^T$. Let $u_n(x) = e^{\lambda_n x} y_n(x)$. Multiplying (2.7) by $e^{\lambda_n x}$ we get

$$u_n(x) = (1, -\lambda_n)^T + \int_x^\infty \kappa(\xi - x, \lambda_n)\, G_n(\xi)\, u_n(\xi)\, d\xi$$

and so

$$\begin{aligned} u_n(x) - u_0(x) &= (0, \lambda_0 - \lambda_n)^T \\ &\quad + \int_x^\infty \kappa(\xi - x, \lambda_n)\, G_n(\xi)(u_n(\xi) - u_0(\xi))\, d\xi + R_n(x), \end{aligned} \tag{2.8}$$

where

$$R_n(x) = \int_x^\infty [\kappa(\xi - x, \lambda_n)\, G_n(\xi) - \kappa(\xi - x, \lambda_0)\, G_0(\xi)]\, u_0(\xi)\, d\xi$$

For $x \geqslant b$ we have by (2.8) and (2.4) that

$$\|u_n(x) - u_0(x)\| \leqslant |\lambda_n - \lambda_0| + \rho_n + \int_x^\infty (\Lambda + \xi - b)\, 2k(\xi)\, \|u_n(\xi) - u_0(\xi)\|\, d\xi. \tag{2.9}$$

where $\rho_n = \max_{x \geqslant b} \|R_n(x)\|$ and $\Lambda = \max\{\lambda_n + 2 : n = 0, 1, 2, \ldots\}$. Applying Gronwall's inequality to (2.9) we obtain, for $x \geqslant b$, that

$$\|u_n(x) - u_0(x)\| \leqslant [(\lambda_n - \lambda_0) + \rho_n] \exp\left(2 \int_b^\infty (\Lambda + \xi - b)\, k(\xi)\, d\xi\right). \tag{2.10}$$

For $x \geqslant b$ we have

$$\begin{aligned} \|R_n(x)\| \leqslant &\int_x^\infty \|\kappa(\xi - x, \lambda_n) - \kappa(\xi - x, \lambda_0)\|\, 2k(\xi)\, \|u_0(\xi)\|\, d\xi \\ &+ \int_x^\infty \|\kappa(\xi - x, \lambda_0)\|\, \|G_n(\xi) - G_0(\xi)\|\, \|u_0(\xi)\|\, d\xi \end{aligned} \tag{2.11}$$

and by (2.5) and (2.4) we have for $\xi \geqslant x \geqslant b$ that

$$\|\kappa(\xi - x, \lambda_n) - \kappa(\xi - x, \lambda_0)\| \leqslant |1 - e^{-2|\lambda_n - \lambda_0|(\xi - b)}|\, |2\Lambda + (\xi - b)| + |\lambda_n - \lambda_0| \tag{2.12}$$

and

$$\|\kappa(\xi - x, \lambda_0)\| \leqslant \Lambda + (\xi - b). \tag{2.13}$$

Using the estimate (2.12) in the first integral on the right side of (2.11), using the estimate (2.13) in the second integral on the right side of (2.11), and then replacing the lower limit of integration of both integrals with b, we have by the dominated convergence theorem that $\rho_n \to 0$ as $n \to \infty$. Thus, by (2.10), $u_n \to u_0$ uniformly on $b \leqslant x < \infty$ as $n \to \infty$.

Suppose $w_n'(x) \to 0$ as $x \to -\infty$ for $n = 1, 2, \ldots$. Let $z_n(x)$ be the solution of (2.1a) satisfying

$$z_n(x) = e^{\lambda_n x}(1 + o(1))$$

as $x \to -\infty$. Then, for $n = 1, 2, \ldots$, we have w_n is a nonzero constant multiple of z_n and hence $W_n(0) = 0$, for $n = 1, 2, \ldots$, where W_n is the Wronskian of z_n and w_n. By the first paragraph of this lemma we have

$$0 = \lim_{n \to \infty} W_n(0) = W_0(0)$$

and hence w_0 is a nonzero constant multiple of z_0. Thus $\lim_{x \to -\infty} w_0'(x) = 0$.

Let $\{c_n\}_{n=0}^{\infty} \subset \mathbb{R} - \{0\}$ be such that $w_n = c_n z_n$ choose x_0 such that $z_0(x_0) \neq 0$. Then

$$c_0 = \frac{w_0(x_0)}{z_0(x_0)} = \lim_{n \to \infty} \frac{w_n(x_0)}{z_n(x_0)} = \lim_{n \to \infty} c_n$$

and hence, since $e^{\lambda_n |x|}(z_n(x), z_n'(x))$ converges uniformly on $-\infty < x \leqslant 0$ to $e^{\lambda_0 |x|}(z_0(x), z_0'(x))$, we have $e^{\lambda_n |x|}(w_n(x), w_n'(x))$ converges uniformly on $-\infty < x \leqslant 0$ to $e^{\lambda_0 |x|}(w_0(x), w_0'(x))$ and, by the first paragraph of this lemma, also on $-\infty < x < \infty$. Hence, since $e^{\lambda_0 |x|}$ $(w_0(x), w_0'(x)$ is bounded, we have $e^{\lambda_n |x|}(w_n(x), w_n'(x))$ and thus $(w_n(x), w_n'(x))$ are uniformly bounded on $-\infty < x < \infty$.

To prove (iv) note that if $w_0(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then $w_0'(x_0) \neq 0$ (for otherwise $w_0 \equiv 0$), and so $w_0(x_1) < 0$ for some x_1. Thus, for all sufficiently large n, $w_n(x_1) < 0$, and the proof of Lemma 1 is complete.

LEMMA 2. *Suppose $g_n, h_n, k_n : \mathbb{R} \to \mathbb{R}$, $n = 0, 1, 2, \ldots$, and $\rho: \mathbb{R} \to [0, \infty)$ are continuous functions, $\int_{-\infty}^{\infty} |x| \rho(x)\, dx < \infty$. g_n, h_n, and k_n converge pointwise on $\mathbb{R}$ to g_0, h_0, and k_0, respectively, as $n \to \infty$, and, for $x \in \mathbb{R}$ and n a nonnegative integer, we have $|g_n(x)|$, $|h_n(x)|$, and $|k_n(x)|$ are each smaller that $\rho(x)$. Let $a \in \mathbb{R}$. Then for each nonnegative integer n the problem*

$$w'' + g_n(x)\, w + h_n(x)\, w' = k_n(x), \tag{2.14a}$$

$$\begin{gathered} w'(-\infty) = 0, \qquad w(-\infty) = a, \\ (w'(\infty) = 0, \qquad w(\infty) = a), \end{gathered} \tag{2.14b}$$

has a unique solution $w_n(x)$. *For each* $b \in \mathbb{R}$, $(w_n(x), w_n'(x))$ *converges uniformly on* $-\infty < x \leqslant b$, $(b \leqslant x < \infty)$, *to* $(w_0(x), w_0'(x))$ *as* $n \to \infty$. *For some continuous functions* $\mu, \nu: \mathbb{R} \to [0, \infty)$ *with*

$$\begin{aligned} \mu(x) &= \mathcal{O}(|x|) \quad \text{as} \quad x \to \infty, \quad (x \to -\infty), \\ &= \mathcal{O}(1) \quad \text{as} \quad x \to -\infty, \quad (x \to \infty). \end{aligned} \tag{2.15a}$$

and

$$\begin{aligned} \nu(x) &= \mathcal{O}(1) \quad \text{as} \quad x \to \infty, \quad (x \to -\infty), \\ &= \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as} \quad x \to -\infty, \quad (x \to \infty), \end{aligned} \tag{2.15b}$$

we have for all nonnegative integers n *and all* x *that* $|w_n(x)| < \mu(x)$ *and* $|w_n'(x)| < \nu(x)$. *For each nonnegative integer* n, $w_n'(\infty) = \lim_{x \to \infty} w_n'(x)$, $(w_n'(-\infty) = \lim_{x \to -\infty} w_n'(x))$, *exists and is finite and* $\lim_{n \to \infty} w_n'(\infty) = w_0'(\infty)$, $(\lim_{n \to \infty} w_n'(-\infty) = w_0'(-\infty))$. *Also, if* $w_n'(\infty)$, $(w_n'(-\infty))$, $= 0$ *for all nonnegative integers* n *then* $w_n(x)$ *converges uniformly on* $-\infty < x < \infty$ *to* $w_0(x)$ *as* $n \to \infty$.

Proof. We will only prove the half of the lemma without parentheses. Under the change of variables $y(x) = w(x), w'(x))^T$ the problem (2.14a), (2.14b) becomes

$$y' = Ay + G_n(x)\,y + K_n(x), \tag{2.16a}$$

$$y(-\infty) = (a, 0)^T, \tag{2.16b}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad G_n(x) = \begin{pmatrix} 0 & 0 \\ -g_n(x) & -h_n(x) \end{pmatrix}, \quad \text{and} \quad K_n(x) = \begin{pmatrix} 0 \\ k_n(x) \end{pmatrix}.$$

First, we show for each nonnegative integer n that the problem (2.16a) (2.16b) has a solution defined for all x. Let n be a fixed nonnegative integer and let $G(x) = G_n(x)$ and $K(x) = K_n(x)$. Choose $x_0 < 0$ such that $\int_{-\infty}^{x_0} (2 + |\xi|)\, \|K(\xi)\|\, d\xi < 1$ and $\int_{-\infty}^{x_0} (2 + |\xi|)\, \|G(\xi)\|\, d\xi < \frac{1}{2}$, where we define the norm of a matrix as the sum of the absolutely values of its entries. We define a sequence of functions $\{y_m: (-\infty, x_0] \to \mathbb{R}^2\}_{m=0}^{\infty}$ as follows. Let $y_0(x) \equiv (a, 0)^T$ and, for $m = 0, 1, 2, \ldots$, let

$$y_{m+1}(x) = (a, 0)^T + \int_{-\infty}^{x} e^{A(x-\xi)} [K(\xi) + G(\xi)\, y_m(\xi)]\, d\xi. \tag{2.17}$$

Suppose, inductively, that $y_m: (-\infty, x_0] \to \mathbb{R}^2$ is continuously differentiable and $\| y_m(x)\| \leqslant 2(1+|a|)$ for $-\infty < x \leqslant x_0$. Then, since $\|e^{Ax}\| = 2 + |x|$, we have for $-\infty < x \leqslant x_0$ that

$$\| y_{m+1}(x)\| \leqslant |a| + \int_{-\infty}^{x_0} (2+|\xi|)\, \|K(\xi)\|\, d\xi + 2(1+|a|)$$

$$\times \int_{-\infty}^{x_0} (2+|\xi|)\, \|G(\xi)\|\, d\xi$$

$$\leqslant |a| + 1 + 2(1+|a|)\tfrac{1}{2} = 2(1+|a|)$$

and clearly y_{m+1} is continuously differentiable on $(-\infty, x_0]$. Since $\int_{-\infty}^{x_0} (2+|\xi|)\, \|G(\xi)\|\, d\xi < \frac{1}{2}$ we have, using (2.17) that

$$\sup_{-\infty < x \leqslant x_0} \| y_{m+1}(x) - y_m(x)\| \leqslant \tfrac{1}{2} \sup_{-\infty < x \leqslant x_0} \| y_m(x) - y_{m-1}(x)\|$$

for $m = 2, 3, \ldots$. Thus, as $m \to \infty$, $y_m(x)$ converges uniformly on $-\infty < x \leqslant x_0$ to a continuous function $y(x)$ with $\| y(x)\| \leqslant 2(1+|a|)$. Thus taking the limit as $m \to \infty$ of both sides of (2.17) we see that

$$y(x) = (a, 0)^T + \int_{-\infty}^{x} e^{A(x-\xi)}\, [K(\xi) + G(\xi)\, y(\xi)]\, d\xi$$

and hence, on $-\infty < x \leqslant x_0$, $y(x)$ is continuously differentiable and satisfies (2.16a), (2.16b). Since (2.16a), (2.16b) is linear $y(x)$ can be extended to a solution of (2.16a), (2.16b) on $-\infty < x < \infty$.

The proof of uniqueness of $y(x)$ is straightforward and will be omitted. For each nonnegative integer n let $y_n(x)$ be the solution of (2.16a) (2.16b). Then

$$y_n(x) = (a, 0)^T + \int_{-\infty}^{x} e^{(x-\xi)A}\, [G_n(\xi)\, y_n(\xi) + K_n(\xi)]\, d\xi \tag{2.18}$$

and hence, for $-\infty < x \leqslant 1$ we have

$$\| y_n(x)\| \leqslant a + \int_{-\infty}^{1} (2+1-\xi)\, \rho(\xi)\, d\xi + \int_{-\infty}^{x} (2+1-\xi)\, 2\rho(\xi)\, \| y_n(\xi)\|\, d\xi.$$

Thus, by Gronwall's inequality, we have for $-\infty < x \leqslant 1$ that

$$\| y_n(x)\| \leqslant \left[a + \int_{-\infty}^{1} (3-\xi)\, \rho(\xi)\, d\xi \right] \exp\left(2\int_{-\infty}^{1} (3-\xi)\, \rho(\xi)\, d\xi \right) \tag{2.19}$$

and, if M is the right side of (2.19), we have for $x \geqslant 1$ that

$$\frac{\|y_n(x)\|}{x} \leqslant \left\|\frac{y_n(1)}{x}\right\| + \frac{\|y_n(x) - y_n(1)\|}{x}$$

$$\leqslant M + \int_1^x \frac{2+x-\xi}{x} \xi 2\rho(\xi) \frac{\|y_n(\xi)\|}{\xi} d\xi + \int_1^x \frac{2+x-\xi}{x} \rho(\xi)\, d\xi$$

$$\leqslant M + \int_1^\infty 2\rho(\xi)\, d\xi + \int_1^x 4\xi\rho(\xi) \frac{\|y_n(\xi)\|}{\xi} d\xi.$$

Thus, by Gronwall's inequality, for $x \geqslant 1$ we have

$$\frac{\|y_n(x)\|}{x} \leqslant \left(M + 2\int_1^\infty \rho(\xi)\, d\xi\right) \exp\left(4\int_1^\infty \xi\rho(\xi)\, d\xi\right). \tag{2.20}$$

So, by (2.19) and (2.20) there is a continuous function. $\mu: \mathbb{R} \to [0, \infty)$. satisfying (2.15a) such hat $\|y_n(x)\| < \mu(x)$ for all x and nonnegative integers n.

Also. for n a nonnegative integer, we have for $x \leqslant -1$ that

$$|xw_n'(x)| = \left| x \int_{-\infty}^x |k_n(\xi) - g_n(\xi)\, w_n(\xi) - h_n(\xi)\, w_n'(\xi)|\, d\xi \right|$$

$$\leqslant \int_{-\infty}^x |\xi|\, \rho(\xi)[1 + 2\mu(\xi)]\, d\xi \tag{2.21}$$

and, for $x \geqslant -1$, that

$$|w_n'(x)| = \left| \int_{-\infty}^x |k_n(\xi) - g_n(\xi)\, w_n(\xi) - h_n(\xi)\, w_n'(\xi)|\, d\xi \right|$$

$$\leqslant \int_{\infty}^x \rho(\xi)[1 + 2\mu(\xi)]\, d\xi < \infty. \tag{2.22}$$

So, by (2.21) and (2.22), there is a continuous function, $\nu: \mathbb{R} \to [0. \infty)$, satisfying (2.15b) such that $|w_n'(x)| < \nu(x)$ for all x and nonnegative integers n.

Let $b \in \mathbb{R}$ be fixed. For $-\infty < x \leqslant b$ we have by (2.18) that

$$\|y_n(x) - y_0(x)\| \leqslant \int_0^b (2 + b - \xi) \|K_n(\xi) - K_0(\xi)\|\, d\xi$$

$$+ \int_{-\infty}^b (2 + b - \xi) \|G_n(\xi) - G_0(\xi)\|\, \mu(\xi)\, d\xi$$

$$+ \int_{-\infty}^x (2 + b - \xi) \|G_0(\xi)\|\, \|y_n(\xi) - y_0(\xi)\|\, d\xi \tag{2.23}$$

and thus, by Gronwall's inequality, we have, for $-\infty < x \leqslant b$, that

$$\| y_n(x) - y_0(x) \| \leqslant \alpha_n \exp \int_{-\infty}^{b} (2 + b - \xi)\, 2\rho(\xi)\, d\xi,$$

where α_n is the sum of the first two integrals on the right side of (2.23). Since, by the dominated convergence theorem, $\alpha_n \to 0$ as $n \to \infty$, we have $y_n(x)$ converges uniformly on $-\infty < x \leqslant b$ to $y_0(x)$ as $n \to \infty$.

Since

$$w_n'(x) = \int_{-\infty}^{x} [k_n(\xi) - g_n(\xi)\, w_n(\xi) - h_n(\xi)\, w_n'(\xi)]\, d\xi$$

we have $w_n'(\infty) = \lim_{x \to \infty} w_n'(x)$ exists and is finite and

$$w_n'(\infty) = \int_{-\infty}^{\infty} [k_n(\xi) - g_n(\xi)\, w_n(\xi) - h_n(\xi)\, w_n'(\xi)]\, d\xi. \tag{2.24}$$

Using (2.24) and the dominated convergence theorem we have $\lim_{n \to \infty} w_n'(\infty) = w_0'(\infty)$.

Suppose $w_n'(\infty) = 0$ for all nonnegative integers n. By the first part of this proof, for each nonnegative integer n, the problem

$$u'' + g_n(x)\, u + h_n(x)\, u' = k_n(x),$$
$$u'(\infty) = u(\infty) = 0,$$

has a unique solution $u_n(x)$ and $u_n(x)$ converges uniformly on $0 \leqslant x < \infty$ to $u_0(x)$ as $n \to \infty$. Let $\varphi_n(x)$ and $\Phi_n(x)$ be the solutions of

$$\varphi'' + g_n(x)\, \varphi + h_n(x)\, \varphi' = 0$$

which satisfy, as $x \to \infty$,

$$\varphi_n(x) = 1 + o(1), \qquad \varphi_n'(x) = o(1),$$
$$\Phi_n(x) = x(1 + o(1)), \qquad \Phi_n'(x) = 1 + o(1).$$

By Lemma 1, $\varphi_n(x)$ converges uniformly on $0 \leqslant x < \infty$ to $\varphi_0(x)$ as $n \to \infty$. Now

$$w_n(x) = u_n(x) + c_n \varphi_n(x) + d_n \Phi_n(x)$$

for some constants c_n and d_n. Since $w_n'(\infty) = 0$ we have $d_n = 0$ for all nonnegative integers n. Choose $x_0 > 0$ such that $\varphi_0(x_0) \neq 0$. Then, for all sufficiently large n, $\varphi_n(x_0) \neq 0$,

$$c_n = \frac{w_n(x_0) - u_n(x_0)}{\varphi_n(x_0)},$$

and thus $c_0 = \lim_{n\to\infty} c_n$. Hence, since $w_n(x) = u_n(x) + c_n\varphi_n(x)$, we have $w_n(x)$ converges uniformly on $0 \leqslant x < \infty$ (and thus, by the first part of this lemma, on $-\infty < x < \infty$) to $w_0(x)$ as $n \to \infty$.

LEMMA 3. *Suppose $g, h\colon \mathbb{R} \to \mathbb{R}$ are continuous, $\int_{-\infty}^{\infty} |xg(x)|\,dx < \infty$, $\int_{-\infty}^{\infty} |xh(x)|\,dx < \infty$, and $\lim_{|x|\to\infty} g(x) = 0$. If $\mu \geqslant 0$, φ_0 is the solution of*

$$\varphi'' + g(x)\,\varphi + h(x)\,\varphi' = \mu^2\varphi, \tag{2.25a}$$

$$\varphi(x) = e^{-\mu|x|}(1 + o(1)) \qquad as \quad x \to -\infty \quad (x \to \infty), \tag{2.25b}$$

and $\varphi_0(x_0) = 0$ for some real number x_0 then the boundary value problem

$$\varphi'' + g(x)\,\varphi + h(x)\,\varphi' = \lambda^2\varphi, \tag{2.26a}$$

$$\varphi'(-\infty) = \varphi'(\infty) = 0 \tag{2.26b}$$

has a positive solution in $\hat{X}$ for some $\lambda > \mu$. Conversely, if (2.26a), (2.26b) *has a positive solution for some $\lambda > 0$ then for all $\mu \in [0, \lambda)$ the solution of* (2.25a), (2.25b) *crosses over the x axis.*

Proof. For $\lambda \geqslant 0$ let $\varphi(x, \lambda)$ be the solution of

$$\varphi'' + g(x)\,\varphi + h(x)\,\varphi' = \lambda^2\varphi, \tag{2.27a}$$

$$\varphi(x) = e^{\lambda x}(1 + o(1)) \qquad \text{as} \quad x \to -\infty. \tag{2.27b}$$

Let $\mu \geqslant 0$ and $\varphi_0(x) = \varphi(x, \mu)$. Suppose $\varphi_0(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Let λ_0 be the supremum of the set Λ of all $\lambda > \mu$ such that $\varphi(x, \lambda)$ has a zero. Since $\varphi_0(x_0) = 0$ we have $\varphi_0'(x_0) \neq 0$, for otherwise $\varphi_0 \equiv 0$. So $\varphi_0(b) < 0$ for some $b \in \mathbb{R}$. By Lemma 1, $\lim_{n\to\infty} \varphi(b, \mu + (1/n)) = \varphi_0(b)$, and hence Λ is not empty and $\lambda_0 > \mu \geqslant 0$. Since, for $\lambda > 0$, $\varphi'(x, \lambda) = \lambda e^{\lambda x}(1 + o(1))$ as $x \to -\infty$, we have $\varphi'(x, \lambda)$ is positive in a neighborhood of $-\infty$. For $\lambda > 0$, let $x_\lambda = \sup\{x\colon \varphi'(\xi, \lambda) > 0 \text{ for } -\infty < \xi < x\}$. If $x_\lambda < \infty$ then $\varphi(x_\lambda, \lambda) > 0$, $\varphi'(x_\lambda, \lambda) = 0$, $\varphi''(x_\lambda, \lambda) \leqslant 0$, and, putting $x = x_\lambda$ in (2.27a), we get

$$(\lambda^2 - g(x_\lambda))\,\varphi(x_\lambda, \lambda) \leqslant 0. \tag{2.28}$$

Thus, for $\lambda^2 > \max_{-\infty < x < \infty} g(x)$, we have $x_\lambda = \infty$ and hence $\lambda \notin \Lambda$. So Λ is bounded above and thus $\lambda_0 < \infty$.

Suppose $\varphi(c, \lambda_0) = 0$ for some $c \in \mathbb{R}$. Then $\varphi'(c, \lambda_0) \neq 0$ and hence $\varphi(d, \lambda_0) < 0$ for some $d \in \mathbb{R}$. Thus, by Lemma 3, $\varphi(d, \lambda_0 + (1/n)) < 0$ for all sufficiently large n, and hence $\varphi(x, \lambda_0 + (1/n))$ has a zero for large n, a contradiction. So $\varphi(x, \lambda_0) > 0$ for all x.

Choose $r > 0$ such that $|g(x)| < \lambda_0^2/4$ for $|x| \geqslant r$. Choose a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \Lambda \cap (\lambda_0/2, \lambda_0]$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$. Since, for $x_\lambda < \infty$, (2.28) holds, we have $x_{\lambda_n} > -r$ and thus $\varphi'(x, \lambda_n)$ and $\varphi(x, \lambda_n)$ are positive for $x \leqslant -r$ and all n.

Since $\varphi(x, \lambda_0)$ is positive for all x we have either (i) $\varphi(x, \lambda_0)$ is in $\hat{X}$, or (ii) $\lim_{x \to \infty} \varphi(x, \lambda_0) = \lim_{x \to \infty} \varphi'(x, \lambda_0) = \infty$. Suppose (ii) holds. Choose $s > r$ such that $\varphi(s, \lambda_0)$ and $\varphi'(s, \lambda_0)$ are both positive. Since, by Lemma 3, $(\varphi(x, \lambda_n), \varphi'(x, \lambda_n))$ converges uniformly on $-r \leqslant x \leqslant s$ to $(\varphi(x, \lambda_0), \varphi'(x, \lambda_0))$, we have, for all sufficiently large n, that $\varphi'(s, \lambda_n) > 0$ and $\varphi(x, \lambda_n) > 0$ for $-r \leqslant x \leqslant s$ and hence also for $-\infty < x \leqslant s$. Since $\lambda_n \in \Lambda$, we have, for all sufficiently large n, that $\varphi'(x_n, \lambda_n) = 0$, $\varphi(x_n, \lambda_n) > 0$, and $\varphi''(x_n, \lambda_n) \leqslant 0$ for some $x_n > s$. Putting $x = x_n$ and $\lambda = \lambda_n$ in (2.27a) we get $(\lambda_n^2 - g(x_n))\, \varphi(x_n, \lambda_n) \leqslant 0$ a contradiction. Thus (i) holds and the proof of the first part of Lemma 3 is complete.

Conversely, suppose $\varphi(x, \lambda)$ is positive and in $\hat{X}$ for some $\lambda > 0$ and $\mu \in [0, \lambda)$. If $\varphi(x, \mu) > 0$ for all x then using Picone's formula for $\varphi(x, \lambda)$ and $\varphi(x, \lambda)$ and the fact that $|\varphi'(x, \lambda)| + |\varphi(x, \lambda)| = \mathcal{O}(e^{-\lambda|x|})$ as $|x| \to \infty$ and $\varphi(x, \mu)\, e^{\mu|x|}$ and $\varphi'(x, \mu)\, e^{\mu|x|}$ are bounded away from zero we get

$$(\lambda - \mu) \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda)}{\varphi(x, \mu)}\, \varphi(x, \lambda)\, \varphi(x, \mu) \exp\left(\int_{\infty}^{\infty} h(t)\, dt\right) dx < 0$$

a contradiction, which completes the proof of Lemma 3.

LEMMA 4. *If* $g, h\colon \mathbb{R} \to \mathbb{R}$ *are continuous and* $\int_{-\infty}^{\infty} (|g(x)| + |h(x)|)\, dx < \infty$ *then for some* $\lambda > 0$ *the solution,* $y(x, \lambda)$, *of*

$$y'' + g(x)\, y + h(x)\, y' = \lambda^2 y, \tag{2.29}$$
$$y(x) = e^{\lambda x}(1 + o(1)) \qquad \text{as} \quad x \to -\infty,$$
$$y'(x) = \lambda e^{\lambda x}(1 + o(1)) \qquad \text{as} \quad x \to -\infty,$$

is positive for all $x \in \mathbb{R}$.

Proof. For each $\lambda > 0$, let $a(\lambda) = \sup\{x : y'(\xi, \lambda) > 0$ for $-\infty < \xi < x\}$. Then, for each $\lambda > 0$, $-\infty < a(\lambda) \leqslant \infty$. Applying Green's formula to $y(x, \lambda)$ and $y(x, \mu)$, where $\lambda > \mu > 0$, we get

$$[y'(x, \lambda)\, y(x, \mu) - y(x, \lambda)\, y'(x, \mu)] \exp\left(\int_{-\infty}^{x} h\right)$$
$$= (\lambda - \mu) \int_{-\infty}^{x} y(\xi, \lambda)\, y(\xi, \mu) \exp\left(\int_{-\infty}^{\xi} h\right) d\xi$$

and hence for all $x \leqslant \min\{a(\lambda), a(\mu)\}$ we have $y'(x, \lambda) > 0$. Thus $a(\lambda) \geqslant a(\mu)$, that is $a(\lambda)$ is an increasing function.

Suppose μ, λ, and ρ are fixed positive numbers such that $a(\mu) < \infty$, $a(\lambda) < \infty$, $\lambda > \mu$, and $\lambda > \max\{|g(x)|^{1/2} : a(\mu) \leqslant x \leqslant a(\mu) + \rho\}$. Then $y(a(\lambda), \lambda) > 0$, $y'(a(\lambda), \lambda) = 0$, $y''(a(\lambda), \lambda) \leqslant 0$ and putting $x = a(\lambda)$ in (2.29) we get $g(a(\lambda)) \geqslant \lambda^2$ and thus, since $a(\lambda) \geqslant a(\mu)$, we have $a(\lambda) > a(\mu) + \rho$. Hence $\lim_{\lambda \to \infty} a(\lambda) = \infty$.

Similarly, if, for $\lambda > 0$, $z(x, \lambda)$ is the solution of

$$\begin{aligned} &z'' + g(x)\, z + h(x)\, z' = \lambda^2 z, \\ &z(x) = e^{-\lambda x}(1 + o(1)) \qquad \text{as} \quad x \to \infty, \\ &z'(x) = -\lambda e^{-\lambda x}(1 + o(1)) \qquad \text{as} \quad x \to \infty, \end{aligned}$$

and $b(\lambda) = \inf\{x : z'(\xi, \lambda) < 0 \text{ for } x < \xi < \infty\}$ then $\lim_{\lambda \to \infty} b(\lambda) = -\infty$.

Choose $\Lambda > 0$ such that $a(\Lambda) > 0$ and $b(\Lambda) < 0$. We claim $y(x, \Lambda) > 0$ for all x. If not then for some $x_0 > 0$ we have $y(x_0, \Lambda) = 0$ and $y(x, \Lambda) > 0$ for $-\infty < x < x_0$. Applying Green's formula to $y(x, \Lambda)$ and $z(x, \Lambda)$ we get

$$y'(x_0, \Lambda)\, z(x_0, \Lambda) \exp\left(\int_{-\infty}^{x_0} h\right) - [y'(0, \Lambda)\, z(0, \Lambda) - y(0, \Lambda)\, z'(0, \Lambda)] \times \exp\left(\int_{x}^{0} h\right) = 0. \tag{2.30}$$

Since $a(\Lambda) > 0$ and $b(\Lambda) < 0$ the left side of (2.30) is negative. This contradiction shows $y(x, \Lambda) > 0$ for all x and completes the proof of Lemma 4.

In the following lemma, if we were to replace the conclusion that $\gamma_1(r)$ and $\gamma_2(r)$ are continuously differentiable for $|r| < r_0$, with the conclusion that $\gamma_1(r)$ and $\gamma_2(r)$ are differentiable for $|r| < r_0$, then the lemma would follow from Magnus [8, Theorem 1 and Corollary 2]. However, to prove $\gamma_1'(r)$ and $\gamma_2'(r)$ are continuous at $r = 0$ requires a separate argument which we omit here since it will appear in a more general setting in [11].

LEMMA 5. *Let $g(x, y)$ be a real valued function which is twice continuously differentiable in an $\mathbb{R}^2$ neighborhood of $(0, 0)$. Suppose $g(0, 0)$, $g_x(0, 0)$, and $g_y(0, 0)$ are all zero, and $B^2 - AC \neq 0$, where $A = g_{xx}(0, 0)$, $B = g_{xy}(0, 0)$, and $C = g_{yy}(0, 0)$. If $B^2 - AC < 0$ then for some $r_0 > 0$ we have $g(x, y) \neq 0$ for $0 < x^2 + y^2 < r_0^2$. If $B^2 - AC > 0$ and l_1 and l_2 are the two lines intersecting at $(0, 0)$ on which the quadratic form $Ax^2 + 2Bxy + Cy^2$ vanishes, then, for some $r_0 > 0$, $T = \{(x, y) : x^2 + y^2 < r_0^2$ and $g(x, y) = 0\}$ is given by the graph of two curves $\gamma_1(r) = (x_1(r), y_1(r))$ and $\gamma_2(r) = (x_2(r), y_2(r))$ defined and continuously differentiable for $|r| < r_0$ with $x_i(r)^2 + y_i(r)^2 = r^2$ and the graph of γ_i tangent to l_i at $(0, 0)$ for $i = 1, 2$. Also $\nabla g(x, y) \neq (0, 0)$ for $(x, y) \in T - \{(0, 0)\}$.*

3. Proofs

In this section we establish the propositions and theorem in Section 1.

Proof of Proposition 1. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\hat{f}(x, y, z, \alpha) = (0, f(x, y, z, \alpha))^{\mathrm{T}}$. Then under the change of variables $u(x) = (y(x), y'(x))^{\mathrm{T}}$ the problem (1.4a), (1.4b) is equivalent to the problem

$$u(x) = (\beta, 0)^{\mathrm{T}} - \int_{-\infty}^{x} e^{A(x-t)} \hat{f}(t, u(t), \alpha)\, dt. \tag{3.1}$$

Let B be a positive real number. Choose $x_0 < 0$ such that $R = \{(x, y, z): x \leqslant x_0, |y| \leqslant B + m, |z| \leqslant m\} \subset \{(x, y, z): x \leqslant x_0, |y| \leqslant m|x|, |z| \leqslant m\}$, and $\int_{-\infty}^{x_0} (2 + |t|) F(t)\, dt$ is smaller than both m and $\frac{1}{2}$. Let $(\alpha, \beta) \in J \times (-B, B)$. Define $\{u_n: (-\infty, x_0] \to \mathbb{R}^2\}_{n=0}^{\infty}$ inductively as follows: $u_0(x) \equiv (0, 0)^{\mathrm{T}}$ and

$$u_{n+1}(x) = (\beta, 0)^{\mathrm{T}} - \int_{-\infty}^{x} e^{A(x-t)} \hat{f}(t, u_n(t), \alpha)\, dt. \tag{3.2}$$

Clearly u_0 is defined for $x \leqslant x_0$ and its graph is contained in R. Assume, inductively, that u_n is defined for $x \leqslant x_0$ and its graph is contained in R. Then, for $x \leqslant x_0$,

$$\|u_{n+1}(x) - (\beta, 0)^{\mathrm{T}}\| \leqslant \int_{-\infty}^{x} (2 + |t|) F(t)\, dt < m$$

(where we define the norm of a matrix as the sum of the absolute values of its entries), and thus u_{n+1} is defined for $x \leqslant x_0$ and its graph is contained in R. Hence, by induction, for all nonnegative integers n, u_n is defined for $x \leqslant x_0$ and its graph is contained in R. For $x \leqslant x_0$ we have, by the mean value theorem, that

$$\begin{aligned}
\|u_{n+1}(x) - u_n(x)\| &\leqslant \int_{-\infty}^{x} (2 + |t|)\, |f(t, u_n(t), \alpha) - f(t, u_{n-1}(t), \alpha)|\, dt \\
&\leqslant \int_{-\infty}^{x_0} (2 + |t|) F(t)\, \|u_n(t) - u_{n-1}(t)\|\, dt \\
&\leqslant \tfrac{1}{2} \max_{t \leqslant x_0} \|u_n(t) - u_{n-1}(t)\|.
\end{aligned}$$

So $u_n(x)$ converges uniformly on $-\infty < x \leqslant x_0$ to a function $u(x)$ whose graph is in R, and taking the limit as $n \to \infty$ of both sides of (3.2) we have, by the dominated convergence theorem, that $u(x)$ satisfies (3.1). Thus u is continuously differentiable on $-\infty < x \leqslant x_0$. If $\hat{u}$ were another solution of

(3.1) then for some $x_1 \leqslant x_0$ we would have that the graph of $\hat{u}$ is in R for $x \leqslant x_1$ and

$$\|\hat{u}(x) - u(x)\| \leqslant \int_{-\infty}^{x_1} (2 + |t|) F(t) \|\hat{u}(t) - u(t)\| dt$$

$$\leqslant \tfrac{1}{2} \max_{t \leqslant x_1} \|\hat{u}(t) - u(t)\|$$

for $x \leqslant x_1$. Thus $\hat{u}(x) \equiv u(x)$ for $x \leqslant x_1$, which shows, for $(\alpha, \beta) \in J \times (-B, B)$, the problem (1.4a), (1.4b) has a unique solution $u(x, \alpha, \beta) = (y(x, \alpha, \beta), y'(x, \alpha, \beta))^T$. Also $u(x, \alpha, \beta)$ is defined at least on $-\infty < x \leqslant x_0$ and, for $x \leqslant x_0$, its graph is contained in R.

Let $(\bar{\alpha}, \bar{\beta})$ and (α, β) belong to $J \times (-B, B)$. Let $\bar{u}(x) = u(x, \bar{\alpha}, \bar{\beta})$ and $u(x) = u(x, \alpha, \beta)$. Then, for $x \leqslant x_0$, we have, by the mean value theorem, that

$$\|\bar{u}(x) - u(x)\| \leqslant |\bar{\beta} - \beta|$$

$$+ \int_{-\infty}^{x} (2 + |t|) F(t) \|\bar{u}(t) - u(t)\| dt + I |\bar{\alpha} - \alpha|,$$

where $I = \int_{-\infty}^{x_0} (2 + |t|) F(t)\, dt$. So, by Gronwall's inequality, for $x \leqslant x_0$, we have

$$\|\bar{u}(x) - u(x)\| \leqslant (|\bar{\beta} - \beta| + I |\bar{\alpha} - \alpha|) e^I$$

and hence $u(x, \alpha, \beta)$ is continuous on $(-\infty, x_0] \times J \times (-B, B)$.

By Lemma 2, for each $(\alpha, \beta) \in J \times (-B, B)$ the problem (1.5a), (1.5b) has a unique solution $w(x, \alpha, \beta)$, and $w(x, \alpha, \beta)$ and $w'(x, \alpha, \beta)$ are both continuous on $(-\infty, x_0] \times J \times (-B, B)$. Let $v(x, \alpha, \beta) = (w(x, \alpha, \beta), w'(x, \alpha, \beta))^T$. Then

$$v(x, \alpha, \beta) = -\int_{-\infty}^{x} e^{A(x-t)} [\hat{f}_u(t, u(t, \alpha, \beta), \alpha)\, v(t, \alpha, \beta)$$

$$+ \hat{f}_\alpha(t, u(t, \alpha, \beta), \alpha)]\, dt.$$

Let $(\alpha_0, \beta_0) \in J \times (-B, B)$. Let $u_0(x) = u(x, \alpha_0, \beta_0)$, $v_0(x) = v(x, \alpha_0, \beta_0)$, and, for $\Delta \neq 0$ and $\alpha_0 + \Delta \in J$, let $u(x, \Delta) = u(x, \alpha_0 + \Delta, \beta_0)$ and $\omega(x, \Delta) = (1/\Delta)[u(x, \Delta) - u_0(x)]$. Then, for $x \leqslant x_0$,

$$\omega(x, \Delta) = -\frac{1}{\Delta} \int_{-\infty}^{x} e^{A(x-t)} [\hat{f}(t, u(t, \Delta), \alpha_0 + \Delta) - \hat{f}(t, u_0(t), \alpha_0)]\, dt$$

$$= -\frac{1}{\Delta} \int_{-\infty}^{x} e^{A(x-t)} \int_0^1 \frac{d}{d\xi} [\hat{f}(t, \xi u(t, \Delta) + (1 - \xi)\, u_0(t), \alpha_0 + \xi\Delta)]\, d\xi dt$$

$$= -\int_{-\infty}^{x} e^{A(x-t)} [\hat{A}_1(t, \Delta)\, \omega(t, \Delta) + \hat{A}_2(t, \Delta)]\, dt, \tag{3.3}$$

where

$$\hat{A}_1(t,\Delta)=\int_0^1 \hat{f}_u(t,\xi u(t,\Delta)+(1-\xi)\,u_0(t),\alpha_0+\xi\Delta)\,d\xi, \tag{3.4}$$

$$\hat{A}_2(t,\Delta)=\int_0^1 \hat{f}_\alpha(t,\xi u(t,\Delta)+(1-\xi)\,u_0(t),\alpha_0+\xi\Delta)\,d\xi, \tag{3.5}$$

and

$$\begin{aligned}
&\|\omega(x,\Delta)-v_0(x)\| \\
&\quad\leqslant \int_{-\infty}^{x} (2+|t|)\,\|\hat{A}_1(t,\Delta)\|\,\|\omega(t,\Delta)-v_0(t)\|\,dt \\
&\quad\quad+\int_{-\infty}^{x_0} (2+|t|)\,\|\hat{A}_1(t,\Delta)-\hat{f}_u(t,u_0(t),\alpha_0)\|\,\|v_0(t)\|\,dt \\
&\quad\quad+\int_{-\infty}^{x_0} (2+|t|)\,\|\hat{A}_2(t,\Delta)-\hat{f}_\alpha(t,u_0(t),\alpha_0)\|\,dt,
\end{aligned} \tag{3.6}$$

and hence, by Gronwall's inequality, we have, for $x\leqslant x_0$, that

$$\begin{aligned}
\|\omega(x,\Delta)-v_0(x)\| &\leqslant A_3(\Delta)\exp\int_{-\infty}^{x_0}(2+|t|)\,\|\hat{A}_1(t,\Delta)\|\,dt \\
&\leqslant A_3(\Delta)\exp\int_{-\infty}^{x_0}(1+|t|)\,2F(t)\,dt,
\end{aligned} \tag{3.7}$$

where $A_3(\Delta)$ is the sum of the last two integrals on the right side of Eq. (3.6). Since $\lim_{\Delta\to 0} u(t,\Delta)=u_0(t)$, we have $\hat{A}_1(t,\Delta)\to\hat{f}_u(t,u_0(t),\alpha_0)$ and $\hat{A}_2(t,\Delta)\to\hat{f}_\alpha(t,u_0(t),\alpha_0)$ as $\Delta\to 0$ and so, by the dominated convergence theorem (note that $\|\hat{A}_i(t,\Delta)\|\leqslant 2F(t)$ for $i=1,2$), we have $A_3(\Delta)\to 0$ as $\Delta\to 0$. Thus, by (3.7), $u_\alpha(x,\alpha,\beta)$ exists on $(-\infty,x_0]\times J\times(-B,B)$ and equals $v(x,\alpha,\beta)$. Since $v(x,\alpha,\beta)$ is continuous on $(-\infty,x_0]\times J\times(-B,B)$, so is $u_\alpha(x,\alpha,\beta)$. Since the second component of $v(x,\alpha,\beta)$ is the derivative with respect to x of its first component we have the second component of $v(x,\alpha,\beta)$ is $(d/dx)\,y_\alpha(x,\alpha,\beta)$. Thus, since the second component of $u_\alpha(x,\alpha,\beta)$ is $(\partial/\partial\alpha)\,y'(x,\alpha,\beta)$, we have $(d/dx)\,y_\alpha(x,\alpha,\beta)=(\partial/\partial\alpha)\,y'(x,\alpha,\beta)$.

Similarly, for each $(\alpha,\beta)\in J\times(-B,B)$ the problem (1.6a), (1.6b) has a unique solution $\hat{w}(x,\alpha,\beta)$, and on the set $(-\infty,x_0]\times J\times(-B,B)$ we have $\hat{v}(x,\alpha,\beta)=(\hat{w}(x,\alpha,\beta),\ \hat{w}'(x,\alpha,\beta))$ is continuous, $u_\beta(x,\alpha,\beta)$ exists, $\hat{v}(x,\alpha,\beta)=u_\beta(x,\alpha,\beta)$, and $(d/dx)\,y_\beta(x,\alpha,\beta)=(\partial/\partial\beta)\,y'(x,\alpha,\beta)$.

From what we have now done, it follows from the standard theorem on smooth dependence on parameters over compact intervals (see [4]) that if $x_1 \in \mathbb{R}$, $(\alpha_0, \beta_0) \in J \times \mathbb{R}$ and $u(x, \alpha_0, \beta_0)$ exists for $x \leqslant x_1$ then for all (α, β) in some neighborhood V of (α_0, β_0) we have $u(x, \alpha, \beta)$ exists for $x \leqslant x_1$ and is continuously differentiable on $(-\infty, x_1] \times V$. Also $(\partial/\partial\alpha)\, y'(x, \alpha, \beta) = (d/dx)\, y_\alpha(x, \alpha, \beta)$, $(\partial/\partial\beta)\, y'(x, \alpha, \beta) = (d/dx)\, y_\beta(x, \alpha, \beta)$ and, for each $(\alpha, \beta) \in V$, $y_\alpha(x, \alpha, \beta)$ and $y_\beta(x, \alpha, \beta)$ satisfy, as functions of x, (1.5a), (1.5b) and (1.6a), (1.6b), respectively, for $-\infty < x \leqslant x_1$.

Let Ω' be the set of all $(\alpha, \beta) \in J \times \mathbb{R}$ such that $u(x, \alpha, \beta)$ exists for $-\infty < x < \infty$ and $\overline{\lim}_{x \to \infty} |y'(x, \alpha, \beta)| < m$. Let $(\alpha_0, \beta_0) \in \Omega'$, $y_0(x) = y(x, \alpha_0, \beta_0)$, $u_0(x) = (y_0(x), y_0'(x))^{\mathrm{T}}$, and $\Delta = (\Delta\alpha, \Delta\beta)$ be such that $(\alpha_0, \beta_0) + \Delta \in J \times \mathbb{R}$. Choose $x_0 > 1$ and $m_0 \in (0, m)$ such that $|y_0'(x)| < m_0$ and $|y_0(x)| < m_0 x$ for $x \geqslant x_0$. Choose $\delta > 0$ such that for $\|\Delta\| < \delta$ we have $|y(x_0, \Delta)| < m_0 x_0$ and $|y'(x_0, \Delta)| < m_0$, where $y(x, \Delta) = y(x, \alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta)$. For $\|\Delta\| < \delta$, let $x(\Delta) = \sup\{x > x_0 : |y'(t, \Delta)| < m \text{ for } x_0 \leqslant t < x\}$. Then, for each Δ with $\|\Delta\| < \delta$, we have $mx - |y(x, \Delta)|$ is positive and increasing on $x_0 \leqslant x < x(\Delta)$ and thus is bounded away from zero on $x_0 \leqslant x < x(\Delta)$, and if $x(\Delta)$ is finite then $y'(x(\Delta), \Delta) = m$. For $x_0 \leqslant x < x(\Delta)$ we have

$$\frac{1}{x} \|u(x, \Delta) - u_0(x)\| \leqslant \frac{1}{x} \|u(x_0, \Delta) - u_0(x_0)\|$$

$$+ \int_{x_0}^{x} \frac{2 + |x - t|}{x} |f(t, u(t, \Delta), \alpha_0 + \Delta\alpha) - f(t, u_0(t), \alpha_0)|\, dt.$$

By the mean value theorem, the integrand of the last integral is less than or equal to $3tF(t)(1/t)\, \|u(t, \Delta) - u_0(t)\| + 3F(t)\, \Delta\alpha$. Hence, by Gronwall's inequality, for $x_0 \leqslant x < x(\Delta)$, we have

$$\frac{1}{x} \|u(x, \Delta) - u_0(x)\| \leqslant C(\Delta) \exp\left(\int_{x_0}^{x} 3tF(t)\, dt\right)$$

$$\leqslant C(\Delta) \exp\left(\int_{x_0}^{\infty} 3tF(t)\, dt\right), \tag{3.8}$$

where

$$C(\Delta) = \frac{1}{x_0} \|u(x_0, \Delta) - u_0(x_0)\| + \Delta\alpha \int_{x_0}^{\infty} 3F(t)\, dt.$$

Thus, for $x_0 \leqslant x < x(\Delta)$, we have, using (3.8), that

$$
\begin{aligned}
&| y'(x, \Delta) - y_0'(x)| \\
&\quad = \left| y'(x_0, \Delta) - y_0'(x_0) - \int_{x_0}^{x} |f(t, u(t, \Delta), \alpha_0 + \Delta\alpha) \ -f(t, u_0(t), \alpha_0)| \, dt \right| \\
&\quad \leqslant | y'(x_0, \Delta) - y_0'(x_0)| + \int_{x_0}^{x(\Delta)} tF(t) \frac{1}{t} \| u(t, \Delta) - u_0(t) \| \, dt \\
&\qquad + \Delta\alpha \int_{x_0}^{x(\Delta)} F(t) \, dt \\
&\quad \leqslant | y'(x_0, \Delta) - y_0'(x_0)| + C(\Delta) \exp \int_{x_0}^{\infty} 3tF(t) \, dt \int_{x_0}^{\infty} tF(t) \, dt \\
&\qquad + \Delta\alpha \int_{x_0}^{\infty} F(t) \, dt,
\end{aligned} \tag{3.9}
$$

and hence, since $C(\Delta) \to 0$ as $\|\Delta\| \to 0$, we have, by taking δ smaller if necessary, that

$$\max_{x_0 \leqslant x < x(\Delta)} | y'(x, \Delta) - y_0'(x)| < m - m_0 \tag{3.10}$$

for $\|\Delta\| < \delta$. So, for $\|\Delta\| < \delta$, we have $x(\Delta) = \infty$, and, Ω' is an open set. Also by (3.8) and (3.9), $(1/x)\, u(x, \Delta)$ and $y'(x, \Delta)$ tend uniformly on $x_0 \leqslant x < \infty$ to $(1/x)\, u_0(x)$ and $y_0'(x)$, respectively. Since $\int_{x_0}^{x} f(t, u_0(x), \alpha_0)\, dt < \infty$ we have $y_0'(\infty) = \lim_{x \to \infty} y_0'(x)$ exists and $| y_0'(\infty)| < m$ and so $\Omega' = \Omega$. Since $y'(x, \Delta) \to y_0'(x)$ uniformly on $x_0 \leqslant x < \infty$ we have $y'(\infty, \alpha, \beta)$ is continuous on Ω.

From what we have now done and parts (b) and (c) of hypothesis (H_1) it is clear that the second paragraph of Proposition 1 holds. Hence, by Lemma 2 and the fact that $y_\alpha(x, \alpha, \beta)$ satisfies (1.5a), (1.5b) we have $\lim_{x \to \infty} (d/dx)\, y_\alpha(x, \alpha, \beta)$ exists and is continuous on Ω. Thus in order to show $(\partial/\partial\alpha)\ y'(\infty, \alpha, \beta)$ exists and is continuous on Ω we need only show, for $(\alpha, \beta) \in \Omega$ that the first equation of (1.7) holds.

Let $(\alpha_0, \beta_0) \in \Omega$. Choose $x_0 \geqslant 1$ and $\delta > 0$ such that $y'(x, \alpha_0 + \Delta, \beta_0) < m$ and $y(x, \alpha_0 + \Delta, \beta_0) < mx$ for $|\Delta| \leqslant \delta$ and $x \geqslant x_0$. For $|\Delta| \leqslant \delta$ let $\omega(x, \Delta) = [u(x, \Delta) - u_0(x)]/\Delta$, where $u(x, \Delta) = u(x, \alpha_0 + \Delta, \beta_0)$ and $u_0(x) = u(x, 0)$. Then, as in (3.3), we have

$$\omega(x, \Delta) = \omega(x_0, \Delta) - \int_{x_0}^{x} e^{A(x-t)} \, [\hat{A}_1(t, \Delta)\, \omega(t, \Delta) + \hat{A}_2(t, \Delta)] \, dt, \tag{3.11}$$

where $\hat{A}_1(t, \Delta)$ and $\hat{A}_2(t, \Delta)$ are given by (3.4) and (3.5). Hence, for $x \geqslant x_0$ and $|\Delta| \leqslant \delta$ we have upon dividing (3.11) by x that

$$\|\omega(x, \Delta)\|/x \leqslant \|\omega(x_0, \Delta)\| + \int_{x_0}^{\infty} 3F(t) \, dt + \int_{x_0}^{x} 6tF(t)[\|\omega(t, \Delta)\|/t] \, dt$$

and so, by Gronwall's inequality,

$$\|\omega(x, \Delta)\|/x \leqslant \left[\|\omega(x_0, \Delta)\| + \int_{x_0}^{\infty} 3F(t)\, dt\right] \exp\left(\int_{x_0}^{\infty} 6tF(t)\, dt\right). \quad (3.12)$$

Since $(\partial/\partial\alpha)\, u(x_0, \alpha_0, \beta_0)$ exists we have $\|\omega(x_0, \Delta)\|$ is bounded on $|\Delta| \leqslant \delta$ and hence, by (3.12), there is a constant $M > 0$ such that $\|\omega(x, \Delta)\| \leqslant Mx$ for $|\Delta| \leqslant \delta$ and $x \geqslant x_0$. For $|\Delta| < \delta$ we have

$$\begin{aligned}
&[y'(\infty, \alpha_0 + \Delta, \beta_0) - y'(\infty, \alpha_0, \beta_0)]/\Delta \\
&\quad = \lim_{x \to \infty} [y'(x, \alpha_0 + \Delta, \beta_0) - y'(x, \alpha_0, \beta_0)]/\Delta \\
&\quad = [y'(x_0, \alpha_0 + \Delta, \beta_0) - y'(x_0, \alpha_0, \beta_0)]/\Delta \\
&\qquad - \int_{x_0}^{\infty} [A_1(t, \Delta)\, \omega(t, \Delta) + A_2(t, \Delta)]\, dt, \qquad (3.13)
\end{aligned}$$

where

$$A_1(t, \Delta) = \int_0^1 f_u(t, \xi u(t, \Delta) + (1 - \xi)\, u_0(t), \alpha_0 + \xi\Delta)\, d\xi$$

and

$$A_2(t, \Delta) = \int_0^1 f_\alpha(t, \xi u(t, \Delta) + (1 - \xi)\, u_0(t), \alpha_0 + \xi\Delta)\, d\xi.$$

Since, for $x \geqslant x_0$ and $|\Delta| < \delta$, $\|A_1(t, \Delta)\| \leqslant 2F(t)$ and $\|A_2(t, \Delta)\| \leqslant F(t)$, we have by letting $\Delta \to 0$ in (3.13) and using the dominated convergence theorem that $(\partial/\partial\alpha)\, y'(\infty, \alpha_0, \beta_0)$ exists and equals

$$\begin{aligned}
&\frac{\partial}{\partial\alpha} y'(x_0, \alpha_0, \beta_0) - \int_{x_0}^{\infty} [f_u(t, u_0(t), \alpha_0)\, u_\alpha(t, 0) + f_\alpha(t, u_0(t), \alpha_0)]\, dt \\
&\quad = \lim_{x \to \infty} \frac{d}{dx} y_\alpha(x, \alpha_0, \beta_0),
\end{aligned}$$

where in the last equation we have used $(\partial/\partial\alpha)\, y'(x_0, \alpha_0, \beta_0) = (d/dx)\, y_\alpha(x_0, \alpha_0, \beta_0)$. By a similar argument $(\partial/\partial\beta)\, y'(\infty, \alpha, \beta)$ exists and is continuous.

The proof of the last paragraph of Proposition 1 involves no techniques other than those used up to this point and so will be omitted.

Proof of Proposition 2. Let $h(x)=f_y(x, y(x, \alpha, \beta), y'(x, \alpha, \beta), \alpha)$ and $k(x)=f_z(x, y(x, \alpha, \beta), y'(x, \alpha, \beta), \alpha)$. Then, by hypothesis (H_1), $h(x)$ and $k(x)$ are continuous, $\int_{-\infty}^{\infty} |xh(x)|\,dx < \infty$, $\int_{-\infty}^{\infty} |xk(x)|\,dx < \infty$, $\lim_{|x|\to\infty} h(x)=0$, and $k(x)$ is bounded. Also $L: X \to Y$ is given by

$$L\varphi = \varphi'' + h(x)\,\varphi + k(x)\,\varphi'.$$

Let $\lambda \in \mathbb{C} - (-\infty, 0]$. First we show that if $(L-\lambda I): X \to Y$ is one to one then it is also onto Y. Let σ and $-\sigma$ be the two square roots of λ with $\alpha = \operatorname{Re}\sigma > 0$. By [4, p. 104, Problem 29] there are unique solutions $\varphi_1(x)$ and $\varphi_2(x)$ of $\varphi'' + h(x)\varphi + k(x)\,\varphi' = \lambda\varphi$ such that

$$\varphi_1(x) = e^{\sigma x}(1+o(1)), \qquad \varphi_1'(x) = \sigma e^{\sigma x}(1+o(1)) \quad \text{as} \quad x \to -\infty, \tag{3.14}$$

$$\varphi_2(x) = e^{-\sigma x}(1+o(1)), \qquad \varphi_2'(x) = -\sigma e^{-\sigma x}(1+o(1)) \quad \text{as} \quad x \to \infty. \tag{3.15}$$

Let $W(x) = \varphi_1(x)\,\varphi_2'(x) - \varphi_1'(x)\,\varphi_2(x)$ be the Wronskian of φ_1 and φ_2. Then $W(x) = W_0 \exp(-\int_{-\infty}^{x} k(\xi)\,d\xi)$ for some constant W_0 and $W_0 \neq 0$ for otherwise $\varphi_1 = (\text{const.})\,\varphi_2$ and hence φ_1 and φ_2 would be in X and thus nontrivial solutions of $(L-\lambda I)\,\varphi = 0$. Since $\int_{-\infty}^{\infty} |k(\xi)|\,d\xi < \infty$, $W(x)$ is bounded and bounded away from zero.

Let $g \in Y$ and define $\varphi: \mathbb{R} \to \mathbb{C}$ by

$$\varphi(x) = \varphi_2(x) \int_{-\infty}^{x} \frac{\varphi_1(\xi)\,g(\xi)}{W(\xi)}\,d\xi + \varphi_1(x) \int_{x}^{\infty} \frac{\varphi_2(\xi)\,g(\xi)}{W(\xi)}\,d\xi. \tag{3.16}$$

Then

$$\varphi'(x) = \varphi_2'(x) \int_{-\infty}^{x} \frac{\varphi_1(\xi)\,g(\xi)}{W(\xi)}\,d\xi + \varphi_1'(x) \int_{x}^{\infty} \frac{\varphi_2(\xi)\,g(\xi)}{W(\xi)}\,d\xi \tag{3.17}$$

and

$$\varphi''(x) = \varphi_2''(x) \int_{-\infty}^{x} \frac{\varphi_1(\xi)\,g(\xi)}{W(\xi)}\,d\xi + \varphi_1''(x) \int_{-\infty}^{x} \frac{\varphi_2(\xi)\,g(\xi)}{W(\xi)}\,d\xi + g(x).$$

Thus φ is C^2 and

$$\varphi'' + h(x)\,\varphi + k(x)\,\varphi' - \lambda\varphi = g. \tag{3.18}$$

We next show φ is in X. Since φ_1 and φ_2 are not in X we have (again by [4, p. 104, Problem 29]) for some nonzero constants c_1 and c_2 that

$$\varphi_1(x) = c_1 e^{\sigma x}(1+o(1)), \qquad \varphi_1'(x) = c_1 \sigma e^{\sigma x}(1+o(1)) \quad \text{as} \quad x \to \infty, \tag{3.19}$$

$$\varphi_2(x) = c_2 e^{-\sigma x}(1+o(1)), \qquad \varphi_2'(x) = -c_2 \sigma e^{-\sigma x}(1+o(1)) \quad \text{as} \quad x \to -\infty. \tag{3.20}$$

By (3.14), (3.15), (3.19), (3.20) we have

$$W(x) = -2c_2\sigma(1+o(1)) \qquad \text{as} \quad x \to -\infty, \tag{3.21}$$

$$= -2c_1\sigma(1+o(1)) \qquad \text{as} \quad x \to \infty. \tag{3.22}$$

We also have

$$\int_{-\infty}^{x} \frac{\varphi_1(\xi)\, g(\xi)}{W(\xi)}\, d\xi = -\frac{g(-\infty)}{2c_2\sigma^2} e^{\sigma x}(1+o(1)) \qquad \text{as} \quad x \to -\infty, \tag{3.23}$$

$$= -\frac{c_1\, g(\infty)}{2c_1\sigma^2} e^{\sigma x}(1+o(1)) \qquad \text{as} \quad x \to \infty, \tag{3.24}$$

and

$$\int_{x}^{\infty} \frac{\varphi_2(\xi)\, g(\xi)}{W(\xi)}\, d\xi = -\frac{c_2\, g(-\infty)}{2c_2\sigma^2} e^{-\sigma x}(1+o(1)) \qquad \text{as} \quad x \to -\infty. \tag{3.25}$$

$$= -\frac{g(\infty)}{2c_1\sigma^2} e^{-\sigma x}(1+o(1)) \qquad \text{as} \quad x \to \infty. \tag{3.26}$$

For example, we will establish (3.24). The proofs of (3.23), (3.25), and (3.26) are similar. By (3.14) and (3.19) there is a positive constant M such that $|\varphi_1(x)| \leqslant Me^{\alpha x}$ for all x. Let $\rho(x)$ satisfy $\varphi_1(x) = c_1 e^{\sigma x}(1+\rho(x))$. Then $\rho(x) = o(1)$ as $x \to \infty$ and

$$e^{-\sigma x} \int_{-\infty}^{x} \frac{\varphi_1(\xi)\, g(\xi)}{W(\xi)}\, d\xi = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = e^{-\sigma x} \int_{-\infty}^{x/2} \frac{\varphi_1(\xi)\, g(\xi)}{W(\xi)}\, d\xi,$$

$$I_2 = e^{-\sigma x} \int_{x/2}^{x} c_1 e^{\sigma\xi} \frac{g(\infty)}{W(\infty)}\, d\xi,$$

$$I_3 = e^{-\sigma x} \int_{x/2}^{x} c_1 e^{\sigma\xi} \frac{\rho(\xi)\, g(\xi)}{W(\xi)}\, d\xi,$$

$$I_4 = e^{-\sigma x} \int_{x/2}^{x} c_1 e^{\sigma\xi} \left(\frac{g(\xi)}{W(\xi)} - \frac{g(\infty)}{W(\infty)}\right) d\xi.$$

Since

$$|I_1| \leqslant e^{-\alpha x} M \left\| \frac{g}{W} \right\|_Y \int_{-\infty}^{x/2} e^{\alpha\xi}\, d\xi \to 0 \qquad \text{as} \quad x \to \infty,$$

$$|I_3| \leqslant e^{-\alpha x} |c_1| \left\| \frac{g}{W} \right\|_Y \left(\max_{x/2 \leqslant \xi \leqslant x} |\rho(\xi)| \right) \int_{-\infty}^{x} e^{\alpha \xi}\, d\xi \to 0 \qquad \text{as} \quad x \to \infty,$$

$$|I_4| \leqslant e^{-\alpha x} |c_1| \left(\max_{x/2 \leqslant \xi \leqslant x} \left| \frac{g(\xi)}{W(\xi)} - \frac{g(\infty)}{W(\infty)} \right| \right) \int_{-\infty}^{x} e^{\alpha \xi}\, d\xi \to 0 \quad \text{as} \quad x \to \infty,$$

and, by (3.22),

$$I_2 = -\frac{c_1\, g(\infty)}{2c_1 \sigma^2} \left[1 - e^{-(\sigma/2)x}\right] \to -\frac{c_1\, g(\infty)}{2c_1 \sigma^2} \qquad \text{as} \quad x \to \infty.$$

We see that (3.24) is correct.

By (3.14), (3.15), (3.19), (3.20), (3.23)–(3.26), we have $\lim_{x \to -\infty} \varphi(x) = -g(-\infty)/\sigma^2$, $\lim_{x \to \infty} \varphi(x) = -g(\infty)/\sigma^2$, and $\lim_{|x| \to \infty} \varphi'(x) = 0$. Thus, by (3.18), $\lim_{|x| \to \infty} \varphi''(x) = 0$, and hence $\varphi \in X$. So we have shown that if $L - \lambda I$ is one to one then it is onto, provided $\lambda \in \mathbb{C} - (-\infty, 0]$.

Next we show $\Sigma \subset \mathbb{R}$. Suppose, to the contrary, $\lambda \in \Sigma - \mathbb{R}$. Then $L - \lambda I$ is not one to one. So for some $\varphi \in X - \{0\}$ we have

$$\varphi'' + h(x)\, \varphi + k(x)\, \varphi' = \lambda \varphi, \qquad \bar{\varphi}'' + h(x)\, \bar{\varphi} + k(x)\, \bar{\varphi}' = \bar{\lambda} \bar{\varphi}$$

applying Green's formula to φ and $\bar{\varphi}$ we get

$$(\lambda - \bar{\lambda}) \int_{-\infty}^{\infty} \varphi(x)\, \bar{\varphi}(x) \exp\left(\int_{-\infty}^{x} k \right) dx = 0$$

and hence $\lambda = \bar{\lambda}$, a contradiction.

Suppose $\lambda \in \Sigma$ and $\lambda > \|h\|_Y$. Then, for some $\varphi \in X - \{0\}$, $(L - \lambda I)\, \varphi = 0$ and $\varphi(x) > 0$ for some $x \in \mathbb{R}$. Since φ decays exponentially to zero as $|x| \to \infty$ we have φ attains its maximum value at some $x_0 \in \mathbb{R}$. Thus $\varphi(x_0) > 0$, $\varphi'(x_0) = 0$, and $\varphi''(x_0) \leqslant 0$ and so $0 \geqslant \varphi''(x_0) = (\lambda - h(x_0))\, \varphi(x_0) > 0$ a contradiction. Thus $\Sigma \cap (0, \infty)$ is bounded above by $\|h\|_Y$.

Next we show zero is the only possible cluster point of $\Sigma \cap (0, \infty)$. Suppose $\{\lambda_n\}_{n-1}^{\infty} \subset \Sigma \cap (0, \infty)$ converges to a positive real number, λ_0. Then for each positive integer n the problem

$$\varphi'' + h(x)\varphi + k(x)\, \varphi' = \lambda_n \varphi, \tag{3.27}$$

$$\varphi(x) = e^{\sqrt{\lambda_n}\, x}(1 + o(1)) \qquad \text{as} \quad x \to -\infty \tag{3.28}$$

has a solution $\varphi_n(x)$ in X. By Lemma 1 the functions $\varphi_n(x)$ are uniformly bounded on $-\infty < x < \infty$ and converge pointwise to a solution $\varphi_0(x)$ in X of

(3.27), (3.28) with $n=0$ as $n\to\infty$. Applying Green's formula to φ_n and φ_0 we obtain

$$0=(\lambda_n-\lambda_0)\int_{-\infty}^{\infty}\varphi_n\varphi_0\exp\left(\int_{-\infty}^{x}k\right)dx \tag{3.29}$$

and by the dominated convergence theorem the integral on the right side of (3.29) converges to $\int_{-\infty}^{\infty}\varphi_0^2\exp(\int_{-\infty}^{x}k)\,dx$. Hence, for all sufficiently large n, $\lambda_n=\lambda_0$. Thus $\Sigma\cap(0,\infty)$ has no positive cluster points.

Since $R(L)\subset\{g\in Y: \lim_{x\to\infty}\int_0^x g(t)\,dt$ exists and is finite$\}$ we have codim $R(L)$ is ∞.

The second to last sentence of Proposition 2 follows from the fourth paragraph of this proof and the fact that the null space of $L-\zeta I$ is at most one dimensional for all $\zeta\in\mathbb{C}$.

The *only if* part of the last statement of Proposition 2 follows from Lemma 3. To prove the *if* part of the last statement of Proposition 2, let $\hat{\lambda}=\max\Sigma$ and suppose $\hat{\lambda}\neq\lambda$. Then $\hat{\lambda}>\lambda\geqslant 0$ and thus, by the second to last sentence of Proposition 2, $\hat{\lambda}$ is an eigenvalue of L. Hence, by the *only if* part of the last sentence of Proposition 2, $L\hat{\varphi}=\hat{\lambda}\hat{\varphi}$ for some positive function $\hat{\varphi}$ in $\hat{X}$. Since $L\varphi=\lambda\varphi$ for some positive function φ in $\hat{X}$ we have by applying Green's formula to φ and $\hat{\varphi}$ that

$$(\lambda-\hat{\lambda})\int_{-\infty}^{\infty}\varphi\hat{\varphi}\exp\left(\int_{-\infty}^{x}k(\xi)\right)dx=0$$

and thus $\lambda=\hat{\lambda}$, a contradiction. So $\lambda=\hat{\lambda}$ and the proof of Proposition 2 is complete.

Proof of Proposition 3. Since $y_\beta(x,\alpha,\beta)$ satisfies (1.6a), (1.6b) we have $y_\beta'(\infty,\alpha,\beta)=0$ if and only if zero is an eigenvalue $L(\alpha,\beta)$. Thus Proposition 3 follows from Proposition 2 and the following statement: If $(\alpha_0,\beta_0)\in\Omega$ and $\{(\alpha_n,\beta_n)\}_{n=1}^{\infty}$ is a sequence in Ω converging to (α_0,β_0) then (i) if $\lambda(\alpha_n,\beta_n)$ is an eigenvalue of $L(\alpha_n,\beta_n)$ for $n=1,2,3,\ldots,$ and $\lim_{n\to\infty}\lambda(\alpha_n,\beta_n)=\lambda_0\in[0,\infty]$ then $\lambda_0=\lambda(\alpha_0,\beta_0)$ and $\lambda(\alpha_0,\beta_0)$ is an eigenvalue of $L(\alpha_0,\beta_0)$; and (ii) if, for $n=1,2,3,\ldots,$ $\lambda(\alpha_n,\beta_n)=0$ and zero is not an eigenvalue of $L(\alpha_n,\beta_n)$ then $\lambda(\alpha_0,\beta_0)=0$.

To prove this statement, let $F_0(x)$ be as in the second paragraph of Proposition 1, let $g_n(x)$ and $h_n(x)$, respectively, be f_y and f_z evaluated at $(x,y(x,\alpha_n,\beta_n),\ y'(x,\alpha_n,\beta_n),\alpha_n)$, and let $I=\int_{-\infty}^{\infty}F_0(x)\,dx$. Then, for sufficiently large n, $|g_n(x)|+|h_n(x)|<F_0(x)$ for $-\infty<x<\infty$.

We first prove (i). Let $\lambda_n=\sqrt{\lambda(\alpha_n,\beta_n)}$. By Proposition 2, the problem

$$\varphi''+g_n(x)\,\varphi+h_n(x)\,\varphi'=\lambda_n^2\,\varphi, \tag{3.30a}$$

$$\varphi(x)=e^{\lambda_n x}(1+o(1))\qquad\text{as}\quad x\to-\infty, \tag{3.30b}$$

$$\varphi'(\infty)=0, \tag{3.30c}$$

has a positive solution $\varphi_n(x)$. Suppose $\lambda_0 = \infty$. By Lemma 4, there exists a positive real number Λ such that the equation

$$\psi'' + e^{2l} F_0(x)\,\psi - F_0(x)\,\psi' = \Lambda^2 \psi$$

has a solution $\psi_0(x)$ which is positive for $-\infty < x < \infty$. Applying Picone's formula to φ_n and ψ_0 we have for sufficiently large n that

$$\int_{-\infty}^{\infty} \left[g_n(x) \exp\left(\int_{-\infty}^{x} h_n\right) - e^{2l} F_0(x) \exp\left(-\int_{-\infty}^{x} F_0\right) \right] dx$$

$$\geqslant \int_{-\infty}^{\infty} \left[\exp\left(\int_{-\infty}^{x} h_n\right) - \exp\left(-\int_{-\infty}^{x} F_0\right) \right] \varphi_n'(x)^2\, dx$$

$$+ \int_{-\infty}^{\infty} \varphi_n(x)^2 \left[\lambda_n^2 \exp\left(\int_{-\infty}^{x} h_n\right) - \Lambda^2 \exp\left(-\int_{-\infty}^{x} F_0\right) \right] dx. \tag{3.31}$$

But, for sufficiently large n, the left side of (3.31) is negative and the right side is positive, a contradiction. So $\lambda_0 \in [0, \infty)$, and thus, by Lemma 1, λ_0 is an eigenvalue of $L(\alpha_0, \beta_0)$ whose eigenspace is spanned by a positive function φ_0. Thus, by Proposition 2, $\lambda_0 = \lambda(\alpha_0, \beta_0)$.

We now prove (ii). By Lemma 3, the solution $\psi_n(x)$ of

$$\psi'' + g_n(x)\,\psi + h_n(x)\,\psi' = 0, \qquad \psi(-\infty) = 1$$

is positive for all x and all positive integers n. By Lemma 1, we have $(\psi_n(x), \psi_n'(x))$ converges pointwise on $-\infty < x < \infty$ to $(\psi_0(x), \psi_0'(x))$ and hence $\psi_0(x)$ is nonnegative, but since $\psi_0(x)$ and $\psi_0'(x)$ cannot simultaneously vanish (for otherwise $\psi_0 \equiv 0$) we have $\psi_0(x)$ is positive for all x. By Lemma 2, for sufficiently large n, $|\psi_n(x)| + |\psi_n'(x)| \leqslant \mu(x)$, where $\mu(x)$ is continuous and $\mu(x) = \mathcal{O}(|x|)$ as $|x| \to \infty$. If (ii) is false, then by Proposition 2, for some $\lambda_0 \in (0, \infty)$ the problem (3.30a)–(3.30c) with $n = 0$ has a positive solution, φ_0. Applying Green's formula to φ_0 and ψ_n we obtain

$$\int_{-\infty}^{\infty} \left[g_0 \exp\left(\int_{-\infty}^{x} h_0\right) - g_n \exp\left(\int_{-\infty}^{x} h_n\right) \right] \varphi_0 \psi_n\, dx$$

$$= \int_{-\infty}^{\infty} \left[\exp\left(\int_{-\infty}^{x} h_0\right) - \exp\left(\int_{-\infty}^{x} h_n\right) \right] \varphi_0' \psi_n'\, dx$$

$$+ \lambda_0^2 \int_{-\infty}^{\infty} \varphi_0 \psi_n \exp\left(\int_{-\infty}^{x} h_0\right) dx. \tag{3.32}$$

Since $|\varphi_0'(x)| + |\varphi_0(x)| = \mathcal{O}(e^{-\lambda_0 |x|})$ as $|x| \to \infty$ we have, by applying the dominated convergence theorem to (3.32) that

$$\lambda_0^2 \int_{-\infty}^{\infty} \varphi_0 \psi_0 \exp\left(\int_{-\infty}^{x} h_0\right) dx = 0$$

a contradiction. Hence $\lambda(\alpha_0,\beta_0)=0$ and the proof of Proposition 3 is complete.

Proof of Proposition 4. To prove Proposition 4 it suffices to show that there is a neighborhood V of (α_0,β_0) such that for $(\alpha,\beta)\in V$ we have (i) $y_\beta'(\infty,\alpha,\beta)<0$ implies $\lambda(\alpha,\beta)>0$; (ii) $y_\beta'(\infty,\alpha,\beta)>0$ implies $\lambda(\alpha,\beta)=0$ and zero is not an eigenvalue of $L(\alpha,\beta)$; (iii) $y_\beta'(\infty,\alpha,\beta)=0$ implies $\lambda(\alpha,\beta)=0$ and zero is an eigenvalue of $L(\alpha,\beta)$.

Actually (i) is true for all $(\alpha,\beta)\in\Omega$. To see this, let $(\alpha,\beta)\in\Omega$ be fixed, and suppose $y_\beta'(\infty,\alpha,\beta)<0$. Then, since $y_\beta(x,\alpha,\beta)$ satisfies (1.6a), (1.6b) we have $y_\beta(x_0,\alpha,\beta)=0$ for some $x_0\in R$. Thus, by Lemma 3, we have $\lambda(\alpha,\beta)>0$.

Since, for all $(\alpha,\beta)\in\Omega$, $y_\beta(x,\alpha,\beta)$ satisfies (1.6a), (1.6b) we have, for all $(\alpha,\beta)\in\Omega$, that $y_\beta'(\infty,\alpha,\beta)=(>)\,0$ implies zero is (is not) an eigenvalue of $L(\alpha,\beta)$. Thus to prove (ii) and (iii) it suffices to prove there is a neighborhood V of (α_0,β_0) such that for $(\alpha,\beta)\in V$ we have (iv) $y_\beta'(\infty,\alpha,\beta)\geqslant 0$ implies $\lambda(\alpha,\beta)=0$. Assuming (iv) is false, there is a sequence $\{\alpha_n,\beta_n\}_{n=1}^{\infty}$ in Ω converging to (α_0,β_0) such that $\lambda(\alpha_n,\beta_n)>0$ and $y_\beta'(\infty,\alpha_n,\beta_n)\geqslant 0$. Thus, by Proposition 2, Lemma 3, and the fact that $y_\beta(x,\alpha_n,\beta_n)$ satisfies (1.6a), (1.6b) we have for some $\xi_n\in(-\infty,\infty)$ that $y_\beta(\xi_n,\alpha_n,\beta_n)=0$ and $y_\beta(x,\alpha_n,\beta_n)>0$ for $x<\xi_n$. Thus $y_\beta'(\xi_n,\alpha_n,\beta_n)<0$. Since $y_\beta'(\infty,\alpha_n,\beta_n)\geqslant 0$, for some $x_n\in(\xi_n,\infty)$ we have $y_\beta'(x_n,\alpha_n,\beta_n)=0$ and $y_\beta(x_n,\alpha_n,\beta_n)<0$. Let

$$\begin{aligned} g_n(x) &= f_y(x, y(x,\alpha_n,\beta_n), y'(x,\alpha_n,\beta_n),\alpha_n) && \text{for}\quad x\leqslant x_n,\\ &=0 && \text{for}\quad x>x_n. \end{aligned}$$

and

$$\begin{aligned} h_n(x) &= f_z(x, y(x,\alpha_n,\beta_n), y'(x,\alpha_n,\beta_n),\alpha_n) && \text{for}\quad x\leqslant x_n,\\ &=0 && \text{for}\quad x>x_n. \end{aligned}$$

By Lemma 1 and the second paragraph of Proposition 1, we have for each $b\in R$ that $y_\beta(x,\alpha_n,\beta_n)$ converges uniformly on $(-\infty,b)$ to $y_\beta(x,\alpha_0,\beta_0)$ as $n\to\infty$. Since (α_0,β_0) is a critical point of S and $y_\beta(x,\alpha_0,\beta_0)$ satisfies (1.6a), (1.6b) we have by Proposition 2 that $y_\beta(x,\alpha_0,\beta_0)$ is bounded away from zero on $-\infty<x<\infty$. Thus $x_n\to\infty$ as $n\to\infty$ and (g_n,h_n) converges pointwise on $-\infty<x<\infty$ to (f_y^0,f_z^0) as $n\to\infty$. Let $\psi_n(x)$ be the solution of

$$\psi''+g_n(x)\,\psi+h_n(x)\,\psi'=0,$$

$$\psi'(-\infty)=0,\qquad \psi(-\infty)=1.$$

Then, for $x\leqslant x_n$, we have $\psi_n(x)=y_\beta(x,\alpha_n,\beta_n)$, and for $x\geqslant x_n$ we have $\psi_n'(x)=0$ and $\psi_n(x)=\psi_n(x_n)<0$. Thus, by Lemma 1, $\psi_n(x)$ converges uniformly on $-\infty < x < \infty$ to $y_\beta(x, \alpha_0, \beta_0)$ as $n \to \infty$. Since $y_\beta(x, \alpha_0, \beta_0)$ is positive and bounded away from zero we have for all sufficiently large n that $\psi_n(x)$ is positive for all x, a contradiction. This proves (iv) and completes the proof of Proposition 4.

Proof of Theorem 1. By Proposition 4, there is a neighborhood V of (α_0, β_0) such that for $(\alpha, \beta) \in V \cap S$ we have $y'_\beta(\infty, \alpha, \beta) > (=, <)\ 0$ if and only if (α, β) is a stable (critical, unstable) point of S. If $\gamma(s)$ is a curve in V satisfying the hypothesis of Theorem 1 and $\omega(s) = y'(\infty, \gamma(s))$ for $0 \leqslant s \leqslant 1$ then $s = 0$ and $s = 1$ are consecutive zeros of ω and hence

$$0 \geqslant \omega'(0)\,\omega'(1) = y'_\beta(\infty, \gamma(0))\, y'_\beta(\infty, \gamma(1))\, \beta'(0)\, \beta'(1)$$

which proves Theorem 1.

Proof of Theorem 2. Since (α_0, β_0) is a critical point of S and expression (1.14) is not zero we have, as pointed out in the paragraph before Theorem 2, that $y'_\beta(\infty, \alpha_0, \beta_0) = 0$, $y'_\alpha(\infty, \alpha_0, \beta_0) \neq 0$, and φ_0 is positive, bounded away from zero, and in X. Thus, by the implicit function theorem and Proposition 1, we have S, in a neighborhood (α_0, β_0), is given by the graph of a continuously differentiable function $\alpha = \alpha(\beta)$ defined in a neighborhood of $\beta = \beta_0$ with $\alpha(\beta_0) = \alpha_0$ and $\alpha'(\beta_0) = 0$.

Let $z(x, \beta) = y(x, \alpha(\beta), \beta)$. It follows from Proposition 1 and the fact that $z'(\infty, \beta) \equiv 0$ that $z_\beta(x, \beta)$ satisfies

$$w'' + f_y w + f_z w' + \alpha'(\beta) f_\alpha = 0,$$
$$w'(-\infty) = w'(\infty) = 0, \qquad w(-\infty) = 1,$$

where the partial derivatives of f are evaluated at $(x, z(x, \beta), z'(x, \beta), \alpha(\beta))$. Since $\alpha'(\beta_0) = 0$ we have $z_\beta(x, \beta_0) = \varphi_0(x)$. Thus $z_\beta(x, \beta_0)$ is bounded away from zero on $-\infty < x < \infty$, and, since, by Lemma 2, $z_\beta(x, \beta)$ tends uniformly on $-\infty < x < \infty$ to $z_\beta(x, \beta_0)$ as $\beta \to \beta_0$, there is an open interval I containing β_0 such that $z_\beta(x, \beta) > 0$ for $-\infty < x < \infty$ and $\beta \in I$.

Let $S^+ = \{\beta \in I : L(\alpha(\beta), \beta)$ has a nonnegative eigenvalue$\}$. For $\beta \in S^+$ let $\varphi(x, \beta)$ be the unique positive solution in $\hat{X}$ of $L(\alpha(\beta), \beta)\,\varphi = \lambda(\alpha(\beta), \beta)\,\varphi$ satisfying

$$\varphi(x, \beta) = \exp(\sqrt{\lambda(\alpha(\beta), \beta)}\; x)(1 + o(1))$$

as $x \to -\infty$. By Proposition 3, $\lambda(\alpha(\beta), \beta) \to \lambda(\alpha_0, \beta_0) = 0$ as $\beta \to \beta_0$ and $\beta \in S^+$. By Lemma 1, $(\varphi(x, \beta), \varphi'(x, \beta))$ converges pointwise to $(\varphi_0(x), \varphi_0'(x))$ as $\beta \to \beta_0$ and $\beta \in S^+$, and for some open interval K, with $\beta_0 \in K \subset I$ we have $(\varphi(x, \beta), \varphi'(x, \beta))$ is uniformly bounded on $(-\infty, \infty) \times (K \cap S^+)$.

Applying Green's formula to $\varphi(x,\beta)$ and $z_\beta(x,\beta)$ we obtain for $\beta \in K \cap S^+$ that

$$\alpha'(\beta)\int_{-\infty}^{\infty} f_\alpha \varphi(x,\beta)\exp\left(\int_{-\infty}^{x} f_z\,dt\right)dx$$

$$= -\lambda(\alpha(\beta),\beta)\int_{-\infty}^{\infty} \varphi(x,\beta)\,z_\beta(x,\beta)\exp\left(\int_{-\infty}^{x} f_z\,dt\right)dx \quad \text{if } \lambda(\alpha(\beta),\beta)\neq 0,$$

$$= 0 \qquad \text{if } \lambda(\alpha(\beta),\beta) = 0, \tag{3.33}$$

where f_α is evaluated at $(x, z(x,\beta),\ z'(x,\beta),\ \alpha(\beta))$ and f_z is evaluated at $(t, z(t,\beta), z'(t,\beta), \alpha(\beta))$. Since, for $\beta \in K \cap S^+$ the integrand of the right side of (3.33) is positive, and, by the dominated convergence theorem and the second paragraph of Proposition 1, the integral on the left side of (3.33) converges to $\int_{-\infty}^{\infty} f_\alpha^0 \varphi_0 \exp(\int_{-\infty}^{x} f_z^0)\,dx$ as $\beta \to \beta_0$, we have by taking K smaller if necessary that

$$-\mathrm{sgn}(\alpha'(\beta))\int_{-\infty}^{\infty} f_\alpha^0 \varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0\right)dx = \mathrm{sgn}\,\lambda(\alpha(\beta),\beta) \tag{3.34}$$

for $\beta \in K \cap S^+$. Thus for $\beta \in K$ we have $(\alpha(\beta),\beta)$ is an unstable (critical) point of S implies the left side of (1.15) is $<(=)\,0$.

It remains to show that the left side of (1.15) is positive when $(\alpha(\beta),\beta)$ is a stable point of S. Suppose to the contrary $\{\beta_n\}_{n=1}^{\infty}$ is a sequence in the domain of $\alpha(\beta)$ converging to β_0 such that the left side of (1.15), with $\beta = \beta_n$, is nonpositive and $L(\alpha(\beta_n),\ \beta_n)$ has no nonnegative eigenvalues. Then, by Lemma 3, $\varphi_n(x) > 0$ for all x, where $\varphi_n(x)$ is the solution of

$$\varphi'' + f_y\,\varphi + f_z\,\varphi' = 0,$$
$$\varphi'(\infty) = 0, \qquad \varphi(\infty) = 1,$$

where the partial derivatives of f are evaluated at $(x, z(x,\beta_n), z'(x,\beta_n), \alpha(\beta_n))$. Thus $\varphi_n'(-\infty) < 0$ and applying Green's formula to $z_\beta(x,\beta_n)$ and $\varphi_n(x)$, and noting by Lemma 2 that $z_\beta'(x,\beta_n) = o(1/|x|)$ as $|x| \to \infty$, we get

$$\varphi_n'(-\infty) = -\alpha'(\beta_n)\int_{-\infty}^{\infty} f_\alpha \varphi_n(x)\exp\left(\int_{-\infty}^{x} f_z\,dt\right)dx. \tag{3.35}$$

By Lemma 2, the second paragraph of Proposition 1, and the dominated convergence theorem, we have the integral on the right side of (3.35) converges to the nonzero number

$$\frac{1}{\varphi_0(\infty)}\int_{-\infty}^{\infty} f_\alpha^0\,\varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0\,dt\right)dx.$$

So, for sufficiently large n, the right side of (3.35) is nonnegative, a contradiction.

For $\beta \in K \cap S^+$, we have, by applying Green's formula to $\varphi(x, \beta)$ and $\varphi_0(x)$, that

$$(\beta-\beta_0)\left[\int_{-\infty}^{\infty} \frac{f_y \exp(\int_{-\infty}^{x} f_z) - f_y^0 \exp(\int_{-\infty}^{x} f_z^0)}{\beta-\beta_0} \varphi(x,\beta)\,\varphi_0(x)\,dx \right.$$
$$\left. - \int_{-\infty}^{\infty} \frac{\exp(\int_{-\infty}^{x} f_z) - \exp(\int_{-\infty}^{x} f_z^0)}{\beta-\beta_0} \varphi'(x,\beta)\,\varphi_0'(x)\,dx\right]$$
$$= \lambda(\alpha(\beta),\beta)\int_{-\infty}^{\infty} \varphi(x,\beta)\,\varphi_0(x)\exp\left(\int_{-\infty}^{x} f_z\right)dx \qquad \text{if } \lambda(\alpha(\beta),\beta)\neq 0,$$
$$= 0 \qquad \text{if } \lambda(\alpha(\beta),\beta) = 0. \tag{3.36}$$

Applying the mean value theorem to the difference quotients in the integrands of the two integrals on the left side of (3.36) and then using the dominated convergence theorem we have the expression in brackets on the left side of (3.36) tends to the nonzero number E. Since the integral on the right side of (3.36) is positive for all $\beta \in K \cap S^+$ we have, by making K smaller if necessary, that

$$\operatorname{sgn}\lambda(\alpha(\beta),\beta) = \operatorname{sgn}((\beta-\beta_0)\,E)$$

for all $\beta \in K \cap S^+$. Hence, using (3.34), we obtain (1.17) for $\beta \in K \cap S^+$.

It remains to show, by making K smaller if necessary, that (1.17) holds for $\beta \in K - S^+$. Suppose to the contrary that $\{\beta_n\}_{n=1}^{\infty} \subset K - S^+$ is a sequence in the domain of $\alpha(\beta)$ converging to β_0 such that (1.17) is false for $\beta = \beta_n$, $n = 1, 2, \ldots$. Then

$$\operatorname{sgn}\left(\alpha'(\beta_n)\int_{-\infty}^{\infty} f_\alpha^0\,\varphi_0 \exp\left(\int_{-\infty}^{x} f_z^0\right)dx\right) \neq -\operatorname{sgn}((\beta_n-\beta_0)\,E). \tag{3.37}$$

Since $\beta_n \in K - S^+$ we have that the left side of (3.37) equals one. Since zero is an eigenvalue of $L(\alpha_0, \beta_0)$ we have $\beta_0 \in S^+$ and hence $\beta_n - \beta_0 \neq 0$ for all n. Thus, by (3.37), $(\beta_n - \beta_0)\,E > 0$. Let $\varphi_n(x)$ be the solution of

$$\varphi'' + f_y\varphi + f_z\varphi' = 0, \qquad \varphi'(-\infty) = 0, \qquad \varphi(-\infty) = 1,$$

where f_y and f_z are evaluated at $(x, z(x,\beta_n), z'(x,\beta_n), \alpha(\beta_n))$. Since $\beta_n \in K - S^+$ we have by Lemma 3 that $\varphi_n(x) > 0$ for all x and hence $\varphi_n'(\infty) \geqslant 0$. Applying Green's formula to φ_n and φ_0 we get

$$
\begin{aligned}
0 &\leqslant \varphi_n'(\infty)\,\varphi_0(\infty)\exp\left(\int_{-\infty}^{\infty} f_z\right) \\
&= -(\beta_n-\beta_0)\left[\int_{-\infty}^{\infty} \frac{f_y \exp(\int_{-\infty}^{x} f_z) - f_y^0 \exp(\int_{-\infty}^{x} f_z^0)}{\beta_n-\beta_0}\,\varphi_n(x)\,\varphi_0(x)\,dx\right. \\
&\quad \left. - \int_{-\infty}^{\infty} \frac{\exp(\int_{-\infty}^{x} f_z) - \exp(\int_{-\infty}^{x} f_z^0)}{\beta_n-\beta_0}\,\varphi_n'(x)\,\varphi_0'(x)\,dx\right].
\end{aligned} \tag{3.38}
$$

Applying the mean value theorem to the difference quotients on the right side of (3.38) and using Lemma 2, the bound on H. and the dominated convergence theorem, we see that the expression in brackets on the right side of (3.38) tends to E as $n \to \infty$, a contradiction which completes the proof of Theorem 2.

Proof of Theorem 3. By (1.7) and the fact that $y_\alpha(x, \alpha_0, \beta_0)$ satisfies (1.5b) we have $y_\alpha(x, \alpha_0, \beta_0) = \mathscr{O}(|x|)$ and $(d/dx)\, y_\alpha(x, \alpha_0, \beta_0) = \mathscr{O}(1)$ as $|x| \to \infty$. Since $y_\alpha(x, \alpha_0, \beta_0)$ satisfies (1.5a), (1.5b) with $(\alpha, \beta) = (\alpha_0, \beta_0)$ we have $v(x) = y_\alpha(x, \alpha_0, \beta_0)$. Since $y_\beta(x, \alpha_0, \beta_0)$ satisfies (1.6a), (1.6b) with $(\alpha, \beta) = (\alpha_0, \beta_0)$ we have $y_\beta(x, \alpha_0, \beta_0) = \varphi_0(x)$ and thus $y_\beta(x, \alpha_0, \beta_0)$ and $(d/dx)\, y_\beta(x, \alpha_0, \beta_0)$ are bounded on $-\infty < x < \infty$. Hence, since hypothesis (H_2) holds, the integrals A, B, and C all converge absolutely.

For $(\alpha, \beta) \in \Omega$ let $g(\alpha, \beta) = y'(\infty, \alpha, \beta)$. By Proposition 1, g is twice continuously differentiable on Ω. Since $y_{\alpha\alpha}(x, \alpha_0, \beta_0)$, $y_{\alpha\beta}(x, \alpha_0, \beta_0)$, and $y_{\beta\beta}(x, \alpha_0, \beta_0)$ respectively satisfy (1.8a), (18b), (1.9a), (1.9b), and (1.10a), (1.10b) with $(\alpha, \beta) = (\alpha_0, \beta_0)$, we have by applying Green's formula to $y_{\alpha\alpha}(x, \alpha_0, \beta_0)$ and $\varphi_0(x)$, $y_{\alpha\beta}(x, \alpha_0, \beta_0)$ and $\varphi_0(x)$, and $y_{\beta\beta}(x, \alpha_0, \beta_0)$ and $\varphi_0(x)$, and using (1.11) and the fact that $\varphi_0(x) = y_\beta(x, \alpha_0, \beta_0)$, that

$$g_{\alpha\alpha}(\alpha_0, \beta_0)\,\varphi_0(\infty)\exp\left(\int_{-\infty}^{\infty} f_z^0\right) = -A,$$

$$g_{\alpha\beta}(\alpha_0, \beta_0)\,\varphi_0(\infty)\exp\left(\int_{-\infty}^{\infty} f_z^0\right) = -B,$$

$$g_{\beta\beta}(\alpha_0, \beta_0)\,\varphi_0(\infty)\exp\left(\int_{-\infty}^{\infty} f_z^0\right) = -C,$$

and thus

$$AC - B^2 = \varphi_0(\infty)^2 \exp\left(2\int_{-\infty}^{\infty} f_z^0\right) \det D^2 g(\alpha_0, \beta_0).$$

Since (1.14) is zero, we have, by the paragraph preceding Theorem 1, that $g_\alpha(\alpha_0, \beta_0) = g_\beta(\alpha_0, \beta_0) = 0$. Thus, by Lemma 5, if $B^2 - AC < 0$ then (α_0, β_0) is an isolated point of S, and if $B^2 - AC > 0$ then, for some neighborhood.

V, of (α_0, β_0), we have (i) holds and $\nabla g(\alpha, \beta) \neq (0,0)$ for $(\alpha, \beta) \in (\gamma_1 \cup \gamma_2) - \{(\alpha_0, \beta_0)\}$. By taking V smaller if necessary, we have, by Proposition 4, that if $(\alpha, \beta) \in \gamma_1 \cup \gamma_2$ then (α, β) is a critical point if and only if $g_\beta(\alpha, \beta) = 0$. Thus, if $(\alpha, \beta) \in (\gamma_1 \cup \gamma_2) - \{(\alpha_0, \beta_0)\}$ then (α, β) is a critical point if and only if $g_\beta(\alpha, \beta) = 0$ and $g_\alpha(\alpha, \beta) \neq 0$ if and only if the tangent line to S at (α, β) is vertical.

It remains only to prove (iii). Let

$$\begin{aligned}\hat{E} = \int_{-\infty}^{\infty} \Big[& f_{yy}^0 v \varphi_0^2 + f_{yz}^0 v' \varphi_0^2 + f_{y\alpha}^0 \varphi_0^2 \\ & + f_y^0 \int_{-\infty}^{x} (f_{zy}^0 v + f_{zz}^0 v' + f_{z\alpha}^0)\, dt \varphi_0^2 \Big] \exp\left(\int_{-\infty}^{x} f_z^0\right) dx \\ & - \int_{-\infty}^{\infty} \int_{-\infty}^{x} (f_{yz}^0 v + f_{zz}^0 v' + f_{z\alpha}^0)\, dt \varphi_0'^2 \exp\left(\int_{-\infty}^{x} f_z^0\right) dx\end{aligned}$$

and let E be given by Eq. (1.16). If $\hat{E} + E\beta_1'(\alpha_0) \neq 0$, we leave to the reader to show by an argument very similar to the one used in the last two paragraphs of the proof of Theorem 2, for $|\alpha - \alpha_0|$ sufficiently small, that $(\alpha, \beta_1(\alpha))$ is a stable (critical, unstable) point of S if and only if

$$(\alpha - \alpha_0)[\hat{E} + E\beta_1'(\alpha_0)] < (=, >)\, 0.$$

Suppose $f_z(x, y, z, \alpha) \equiv 0$ in $D \times J$. Then $\hat{E} = B$ and $E = C$. Since γ_1 is tangent to l_1 at (α_0, β_0) we have

$$A + 2B\beta_1'(\alpha_0) + C\beta_1'(\alpha_0)^2 = 0 \tag{3.39}$$

and solving (3.39) for $\beta_1'(\alpha_0)$ we get

$$\beta_1'(\alpha_0) = (-B \pm \sqrt{B^2 - AC})/C$$

Thus $B + C\beta_1'(\alpha_0) = \pm \sqrt{B^2 - AC} \neq 0$ and the proof of Theorem 3 is complete.

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